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TOWARDS A CRITICAL MATHEMATICS EDUCATION

ABSTRACT. To illustrate aspects of critical mathematics education a project involving 14–15 years old students is described. Mathematics education can be organized so as to develop different types of knowing: mathematical knowing, which can be associated with skills developed in traditional teaching; technological knowing, which can be associated with a competence in mathematical model building; and reflective knowing, which can be seen as a competence in evaluating applications of mathematics. The thesis discussed says that if mathemacy should be developed as a competence of importance in a critical education, it must integrate mathematical, technological as well as reflective knowing. Via the description of the project, a possible educational meaning is given to this thesis. Especially, it is discussed what it could mean to involve students in reflections about mathematics as a tool for technological design.[1]

INTRODUCTION

When used in the connection "mathematics education and technology" one easily associates technology with computers in the classroom (or, by a broader interpretation, technology means equipment used in mathematics education). Guiding questions for curriculum planning then become: How to use a technology to make it easy for the students to grasp mathematical ideas? How to use computers in the service of mathematics education? However, that needs not to be the only way to interpret the concept "technology". By "technology" we could also make a reference to technological megastructures of society. For instance, in *The Technological Society* the French sociologist Jacques Ellul extends the perspective and sees technology as a fundamental social structuring principle. Technology concerns all aspects of social life. Our whole civilisation becomes a technological reconstruction, and an attempt could be made to relate mathematics education to that concept of technology.

Now guiding educational questions could become: How to make students aware of the technological impact on society? How to make students aware of the role of mathematics as part of a technological development? How to get an idea of the basic conditions for living in a highly technological society? How to reflect upon the technological culture? Such questions may reveal an opening to critical education, and also to critical

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mathematics education. (Coming from Denmark, my perspective is that of staying in the outskirts of the power centres but at the same time being in the middle of a highly technological development; that naturally influences my conception of critical education.)

Obviously, focussing on the role of mathematics as part of a technological development presupposes that mathematics is "doing something" to society. I shall use the formulation that *mathematics is formatting society,* or that mathematics has a formatting power. Mathematics is seen as a basic principle for technological design. This shall be understood along formulations used by Philip Davis and Ruben Hersh in *Descartes' Dream.* They make a distinction between mathematics used in descriptions, predictions and prescriptions. By a prescriptive use mathematics leads to some sort of human or technological action:

> We are born into a world with so many instances of prescriptive mathematics in place that we are hardly aware of them, and, once they are pointed out, we can hardly imagine the world working without them. Our measurements of space and mass, our clocks and calenders, our plans for buildings and machines, our monetary system, are prescriptive mathematizations of great antiquity. To focus on more recent instances ... think of the income tax. This is an enormous mathematical structure superposed on an enormous pre-existing mathematical financial structure. ... In American society, there are plentiful examples of recent and recently reinstated prescriptive mathematization: exam grades, IQ's, life insurance, taking a number in a bake shop, lotteries, traffic lights ... telephone switching systems, credit cards, zip codes, proportional representation voting ... We have prescribed these systems, often for reasons known only to a few; they regulate and alter our lives and characterize our civilization. They create a description before the pattern itself exists." (Davis and Hersh, 1988, pp. 120–121).

Mathematics can be seen as part of a process of system development, which is a way of putting computing into practice. Systems can be developed for marketing, business, economic management etc., and mathematics becomes a principle for design.[2] If we "subtract" the mathematical competence from our highly-technological society, what is then left? The residue could hardly have much in common with our present society. That means that mathematics has become a part of our culture. Naturally, mathematics *as such* cannot exercise much formatting power, to maintain that would mean to try to defend a pure idealistic attitude in philosophy. In order to make some sense of "mathematics has a formatting power" we have to investigate the impact of the mathematical research paradigm on technology and society.[3] However, I shall not try to develop this analysis but just summarize it by the thesis about the formatting power of mathematics. Therefore, the thesis cannot be looked upon as a justified statement. It remains a thesis to be used as background for my interpretation of mathematics education.[4]

CRITICAL EDUCATION

In 1966 Theodor Adorno published the article "Erziehung nach Auschwitz"[5] ("Education after Auschwitz"), which by many has been conceived as a cornerstone in critical education.[6] The article also contains several odd formulations, which for instance reveal Adorno's bias against rural perspectives. Nevertheless, it has highlighted the importance of education provocatively – not only as an effort to deliver information – but also as part of a cultural and political struggle. Adorno puts education in the position of a social and political force by maintaining that the very first demand to an education is that a new Auschwitz will never happen again. Even if it is an illusion that education can prevent social and political catastrophes, education cannot set aside a responsibility to try to fight for human rights. If this responsibility is overlooked, education runs a risk of being a follower.

When used in the expression "critical education", the meaning of "critical" needs explaining. In fact the term "critique" has a long history starting in Ancient Greece. An important development took place during the Enlightenment, a further development is found in *Critique of Pure Reason* in which Immanuel Kant tries to explicate the general conditions for obtaining (certain) knowledge. The term got a more radical interpretation in Critical Theory, as developed by Max Horkheimer, Theodor Adorno, Herbert Marcuse and others belonging to the Frankfurt School. Also the work of Jürgen Habermas has to be mentioned.[7]

As a most incomplete summary of a "history of critique" we can say that critique has to do with: (1) An investigation of conditions for obtaining knowledge, (2) an identification of social problems and evaluation making, and (3) a reaction to problematic social situations. In other words, the concept of critique points out demands about selfreflections, reflections and reactions. Those demands have been present in the development of critical education as it took place in the German and Scandinavian tradition beginning in the late 60ies.[8]

Critical education has manifested itself in a variety of catchwords: problem orientation, project organization, *Fachkritik*,[9] subjective relevance, interdisciplinarity, emancipation, etc. Some inspiration is found in Critical Theory,[10] but the concept of critical education has many different sources.[11] Paulo Freire, for instance, developed his ideas quite independently of Critical Theory.[12]

The most general and uniting idea is: *If education, as both a practice and a research, should be critical it must discuss basic conditions for obtaining knowledge, it must be aware of social problems, inequalities, suppression etc., and it must try to make education an active progres-*

sive social force. A critical education cannot be a simple prolongation of existing social relationships. It cannot be an apparatus for prevailing inequalities in society. To be critical, education must react to social contradictions.

Naturally, a consequence of this is that in a society with no actual or potential conflicts, a society with everything put in the "right order", critical education will be superfluous. If Plato was right in identifying the very best society, it would not be necessary to invent a critical education. We shall not expect a section about critical education in *The Republic*. In that case education could be a simple introduction to the life and institutions of the "ideal" society. To put it differently: Not to subscribe to critical education means either to accept the social situation as preferable (not necessary as perfect), or to maintain that education does not have a role to play as a critical social force.

LITERACY AND MATHEMACY

In critical education the discussion of "literacy" has played a major role, especially caused by the work of Paulo Freire, who developed the political dimension of education from that term. A basic assumption could be stated as follows: – Literacy is a necessary condition in today's society for informing people about their obligations, and for people to be used in essential work-processes. However, literacy can be used for the purpose of empowerment, because it can be a means to organize and reorganize interpretations of social institutions, traditions and proposals for political reforms.[13]

The competence described by "literacy" contains a conservative element. To be literate is a necessary condition for being a part of the workforce.
But literacy also opens for a reaction to contradictory aspects of social life. Literacy is, so to say, a double edged sword. If that formulation makes sense, it also makes sense to conceive "literacy" as a competence which could be developed as part of a critical education.

Could "mathemacy" be substituted for "literacy" in the previous formulations?[14] By mathemacy we may initially understand an ability in calculating and using mathematical and formal techniques. Does it make sense to say: – Mathemacy is a necessary condition in today's society for informing people about their obligations, and for people to be used in essential work-processes? Well, it makes sense, and in fact it has been quite common to perceive mathematics education as an essential preparation of the coming workforce and, in a wider perspective, as essential to economical growth. But what about: – Mathemacy can be used for the purpose of empowerment, because it can be a means to organize and reorganize interpretations of social institutions, traditions and proposals for political reforms?

Does mathemacy constitute an important competence in interpreting features of a highly technological society? Does mathemacy belong to the "double-edged-sword-competences"? In other words: Is critical mathematics education possible?[15] Such questions cannot be answered by an example, but an example can be developed with those questions in mind.

AN EXAMPLE

In 1988 the Danish Research Council for the Humanities decided to set up the general research initiative "Mathematics Education and Democracy in Highly Technological Societies". The overall intention was to discuss mathematics education as part of a democratic endeavour in a highly technological society.[16] My present work and the experimental work I am going to describe make up parts of that project.

It need not be true that mathematics has a formatting power. However, if it makes sense to look at mathematics as a constituent of a variety of social phenomena, then it is not contradictory to suppose that mathemacy can become a means for "organizing and reorganizing interpretations of social institutions, traditions and proposals for political reforms". However, even if this formulation has some analytical bearing, it need not contain much educational meaning in the sense that it could be operationalized as an educational practice.

Is it possible in elementary mathematics education to make any sense of seeing mathematics as a principle for design? Is it possible to create situations in which the students are using competences, developed in mathematics education, in making interpretations of social phenomena? Is it possible to involve students in a process of system development in order to make them understand something about conditions of mathematical based management? These were main questions in planning the project "Family Support in a Micro-Society".

The teacher, Henning Bødtkjer, and I tried to develop an example of using mathematics as a tool for organizing a little part of a social reality. The project had to do with child benefit.[17] On the basis of a description of a micro-society the task of the students was to distribute a certain amount of money as child benefit. We decided to talk about "family support" instead in order to make the problem more complex, however. The students involved, 20 in all, were between 14 and 15 years old; the school was Klarup Skole near Aalborg.

It was not intended by means of the project to "prove" that critical mathematics education has advantages, which makes it preferable to other approaches to mathematics education. The aim was much more limited. We tried to point out some possible *meanings* of critical mathematics education. Does such an approach make sense in terms of what has been said about "mathemacy" as a parallel to "literacy"?

I will summarize the project in eight units. Each unit may contain more than one lesson.

Unit 1

The general ideas of child benefit and financial support for families were discussed. The students were divided into five groups, in most cases two boys and two girls. They had to work in the same groups for the rest of the project,which lasted two weeks. The groups were asked to make a description of a few imaginary families. They had to include some specific information: the structure of the family, the number of children, their ages, the income of the parents, etc. Besides this, they were welcome to make up whatever else information they found of interest. The described families, 24 in all, made up the Micro-Society which was referred to during the rest of the project.[18] I quote three of the descriptions:

Family 3: The Westergaard family lives in a flat at the Vesterbro district in Copenhagen. Laila is 5 years old and plays ice hockey every Monday and Wednesday. She gets 5 Dkr. pocket money a week, because she is 5 years old. She is saving money for a new ice hockey stick. Esben is 10 years old and is taking saxophone classes every Tuesday. Each Friday he does folk dancing. Esben gets 10 Dkr. pocket money a week, because he is 10 years old. Then there is Torkild. He is 14 years old. He does not have any spare time occupations. He is chasing girls. He gets 50 Dkr. pocket money a week. Finally, there is big sister Pia. She is 21 years old. She has left home and lives together with a guest worker. Pia does not work, so she is on the dole, and the money she gets does not go very far. The father of the children is called Eskild. He is 56 years old and works as a stores clerk. He makes 14,493 Dkr. a month. The mother is called Christina, and she is 37 years old. She works as an assistant in a department store. She makes 12,019 Dkr. a month. Christina has a knitting club together with some friends on Tuesdays. She has just bought a fur coat for 5000 Dkr. She spent her savings in this way. Eskild's old mother died recently, and she left them 20,000 Dkr., and with this money they paid their debts. 5000 Dkr. were left.

Family 12: Lars is a little boy who lives in the country. His father is a pig breeder and makes 10,000 Dkr. a month, and his mother works at a school and makes 10,000 Dkr. a month. They have difficulties in managing after the tax has been paid, and Lars does not get the horse which he has been wanting for the past two years. Everybody in his class (the seventh) has got a computer, but Lars has got a new front tyre for his 10 year old bike. Everybody gets new clothes all the time, but Lars gets the clothes which his father used in his younger days. Several times his father's farm has been on the point of being sold by order of the court, but each time his father has succeeded in preventing it at the eleventh hour. He inherited some money, but he spent it on renovation of the farm two years ago.

Family 21: Mogens Sørensen is looking forward to returning to his family after a hard day's work as manager of the local refuse disposal plant called Reno Nord. His monthly salary totals 24,482 Dkr., but he is likely to get a raise in the near future, as he is soon going to celebrate his 25th anniversary in the company. On his way home he buys flowers for his wife Inger Sørensen as they have been married for eleven years. Inger is assistant in the perfume department of the department stores called Salling. Her monthly salary totals 11,322 Dkr. Tonight they are going to have roast pork for supper which is the favourite dish of Mogens. This morning their child of the marriage, Camilla of 10 years, prepared breakfast, and in addition she bought a box of chocolates from her weekly allowance, totalling 25 Dkr. As Mogens is coming across the cement works, Aalborg Portland, he is passing his son Joachim of 17 years from his previous marriage. Each month Mogens pays an amount of money to him. Mogens is waving, but Joachim does not see him as he is passing in the opposite direction on his moped. Mogens hopes that Camilla has returned from the riding school called Annebergvejens Rideskole, where her horse is stabled. It is expensive, Mogens thinks, but when Camilla is getting older she has promised to find a job herself, so that she can pay half of the 800 Dkr. a month. Finally Mogens turns into the drive-way of his house at Hasserisvej.

Unit 2

The students had to discuss the standards according to which they wanted to distribute a fixed amount of money: 240,000 Dkr. This amount made it possible to give family support in the Micro-Society at a similar level as in Denmark in general. Each group could think of themselves as the authorities of a District, and they had to decide about the distribution of family support. The students had also to specify what data they needed in order to complete the necessary calculations based on their guidelines.

Many standards were suggested, but also a great willingness to make changes were revealed; one rather common reason was that the suggested principles for making the distribution might be too difficult to handle in practice. So already at this initial stage the students exercised a certain degree of self-censorship.

Unit 3

The students received a nicely typed edition of their descriptions, the "Family Circle". The teacher and I found it important not to distribute the descriptions in the students' original handwritings but to provide them with a nice layout. The students spent some time in reading the stories seeing how their own story appeared. In two of the groups one of the students read aloud from the Family Circle, while the rest were listening and making comments.

Henning Bødtkjer and I had added a few family stories, because it turned out that the students seemed to be fascinated by well-to-do families with good jobs and "happy" children. Some exceptions existed of course, but Henning Bødtkjer and I added a few more.

Unit 4

During the study of the Family Circle the students realized that it was difficult to keep an overview of the Micro-Society. This made it natural for the teacher to suggest that data could be installed on a computer. The groups therefore had to read carefully through the Family Circle in order to pick out the information which was relevant to their standards of distribution. The students' task was to create a data base. However, in order to install the information it was necessary to make interpretations of what was told in the Family Circle. It became obvious also that the descriptions in the Circle were insufficient in some respect.

The groups had to make some decisions about the missing information. The obvious way of doing this was to address the student who had written about the family; at least he or she would know. Often the students were rather fast in making a decision: – I see, the age of the child is missing. Anyway, the family lives in number 13, so the age may be 13 too!

Unit 5

The groups had to specify an algorithm for the distribution. As a consequence, the terminology had to be semi-mathematical or semi-algorithmic. In this phase it became obvious that the general and initially formulated guidelines were insufficient and imprecise more often than not. If the task was to do calculations, new demands concerning the formulation of the guidelines were raised. (If the group found the parents' income of relevance, the descriptions had to include for instance whether they wanted to adjust the family support to a "step-scale" or to use proportionality.) The data base enabled the students to get new information easily, for instance: How many families have a total income below a certain sum of money?[19]

One way to control whether an algorithm was specified by a group or not, is to imagine that the group could get an assistant in making their calculations. If they were able to explain to that newcomer how to do the calculations without having to explain all the general principles for the distribution, the group might have specified an algorithm. This definition of algorithm was understandable to the students, and acceptable as well.

The applications of the algorithms became more fancy than we had expected. The shift from a mathematical model to an algorithm were not made obvious in the project. One strategy was just to start with some arbitrary sum of money to be given to a single child, and then to find out how much money was left, and thereupon to find out how to distribute some more money.

Unit 6

The groups had to complete a list of the money received by each of the 24 families in the District. It became rather interesting to compare the five Districts. A list was made on the blackboard. The families were taken one by one, and while the description of a family was read aloud by the teacher, the different groups had to write on the blackboard what they had suggested for that family. That made it possible to make a direct comparison. How is that family treated in the different Districts? And immediately the question could be raised: What is the reason that it is treated so differently? What social policy is dominating in the different Districts? Does it reveal a conservative or a labour dominance? A common comment put forward by the students in this discussion was: – We agree that a difference exists, but

TABLE I

The suggested distributions

Family	Group 1	Group 2	Group 3	Group 4	Group 5
1	13,907.00	20,940.00	13,333.00	9,880.00	10,077.00
2	8,814.00	8,381.00	13,333.00	6,000.00	5,043.00
3	18,314.00	18,069.00	19,999.00	12,000.00	7,543.00
4	46,128.00	42,161.00	46,665.00	19,600.00	35,077.00
5	3,000.00	0.00	0.00	0.00	2,543.00
6	8,814.00	18,069.00	6,666.00	11,800.00	15,077.00
7	18,314.00	12,571.00	19,999.00	8,800.00	7,543.00
8	32,221.00	20,052.00	33,332.00	14,600.00	12,543.00
9	9,500.00	6,023.00	6,666.00	4,000.00	5,077.00
10	8,814.00	8,381.00	13,333.00	8,000.00	5,043.00
11	6,000.00	0.00	0.00	0.00	5,043.00
12	4,707.00	6,023.00	6,666.00	5,000.00	5,077.00
13	8,814.00	20,940.00	6,666.00	6,000.00	10,077.00
14	6,000.00	0.00	0.00	0.00	0.00
15	3,000.00	0.00	0.00	0.00	2,543.00
16	0.00	5,755.00	0.00	1,800.00	5,043.00
17	0.00	0.00	0.00	0.00	0.00
18	8,814.00	12,046.00	13,333.00	8,332.00	5,043.00
19	6,000.00	5,755.00	0.00	4,000.00	5,043.00
20	6,000.00	5,755.00	0.00	1,800.00	5,043.00
21	4,407.00	12,046.00	6,666.00	2,800.00	2,543.00
22	6,000.00	0.00	0.00	1,800.00	5,043.00
23	8,814.00	5,755.00	13,333.00	5,800.00	5,043.00
24	8,814.00	8,381.00	13,333.00	1,000.00	10,077.00
Total	245,196.00	237,103.00	233,323.00	133,012.00	171,184.00

anyway we have made the calculations correctly. The list of the suggested distributions from the five Districts is shown in Table I.

It is interesting to compare the suggestions. There seems to be some general agreement. Family 4 got the most in all Districts, that was the family of the foreign worker with many children and a low income. It was also agreed that family 17 should not receive anything, in fact that "family" consisted of a single old man going to celebrate his 90th birthday together with his children and grandchildren.

Unit 7

The groups had to write a letter to some of the families statuig how much money they would get, and how the District calculated the sum. This task could force the students to try to recapitulate what they in fact were doing when they made their calculations.

Unit 8

We planned to return to a general discussion of the standards of the distribution by asking the students to decide upon one final system of distribution. At that stage the discussion could be dominated by the conflict between what was conceived as fair and what was possible to manage from a technical point of view. This conflict is essential in the discussion of the formatting power of mathematics. But the discussion did not take place. A school holiday was getting closer, and the students' attention was drawn in a new direction.

The more principal aspects of the formatting power of mathematics were touched upon by comparing with other sorts of social regulations in which the conflict between what is possible to manage and what is reasonable and fair are contradicting each other.

COMMENTS ON THE PROJECT

As mentioned earlier the students had to discuss and find out about their general guidelines for the distribution. One of the groups (Group 4) had formulated the following principles: (1) The families who earn a lot should receive less than families with a small income. (2) If the children are older than 18, the family should not receive anything. (3) If a family has only one child and earns a lot, it should not receive any support, but if it has more than one child, it should receive an amount according to their needs. (4) Families with quite young or older children should receive more support (young children and teenagers were supposed to be the most "expensive"). (5) A single parent should receive some extra. On this basis the group listed the following set of necessary data: (1) the number of children in the family, (2) the income of the parents, (3) the age of the children, (4) whether the family consisted of a single parent or a couple, and (5) whether the parents were divorced or separated, and whether the children got support from the other parent or not.

One of the groups, Group 2, made calculations, which could illustrate the *ad hoc* nature of the used algorithms. First they stated that the total amount of money was 240,000 Dkr. and that 45 children lived in the society, later, after some extra countings, the number was changed to 51. From that they calculated the percentage each child could receive – that was 1.961%. Thereupon they stipulated five intervals of income: above 650,000; 650,000–480,000; 479,999–320,000; 319,999–200,000; 199,999–0. They reviewed their principles: families who earn more than 650,000 Dkr. shall not receive any support, and families who earn less should receive more than families who earn more. Then they graduated the percentage each child should receive depending on the income interval of the family. The five percentages became: 0%, 0.980%, 1.307%, 1.961% and 3.945%. They got these figures by taking the average percentage and modifying the figures according to the income levels. After some calculations the group found that 53,336 Dkr. was still left. That money they also wanted to distribute. To do so the group used a bit different percentages: 0%, 0.987%, 1.975%, 2.469% and 3.457%. After this calculation still 2,897 Dkr. were left. Since time was gone the District kept that money. This group became a bit surprised that their first calculation did not come out exactly. They had thought of their original procedure, not as an *ad hoc* way of making the calculations but as an exact one.[20] In general the suggestions as given by the different groups did not add up to the total 240,000 Dkr. In Group 4 the difference is rather high, but as they said: – Our District always has to save money if possible.

Finally, the students were asked whether they would have distributed the money differently or not, if they did not have to use a mathematical model. Some of the comments were: – Well, I think we would have distributed the money differently, if we did not have had any model. I think we would have taken a few things more into account. I think we still would have taken into consideration income, number of children, and the children's age, as we did in the model. However, perhaps also the total assets of the family. Perhaps we would have made a more fine-graded scale. For instance, I would have given Family 13 a bit more than 8,814 Dkr. as we actually did (From Group 1). – Our distribution is based on the model we made. I do not think we would have distributed the money differently. I wouldn't. I think the money has been distributed in a fair way. ... In fact I think that only parents who have no possibilities to bring up their children should receive money. The others do not need the money (From Group 3). – In one way I think I would have done something differently, if I had a complete family description. Something more could have been taken into account, for instance if divorced parents receive other forms of support (From Group 5).

MATHEMACY AND REFLECTIVE KNOWING

Let me make a distinction between three types of knowing towards which a mathematics education can be oriented: (1) *Mathematical knowing*, which refers to the competence normally understood as mathematical skills including competences in reproducing theorems and proofs, as well as mastering a variety of algorithms. This competence is in focus in traditional mathematics education, and its importance has especially been stressed by the structuralistic or "new math" movement. (2) *Technological knowing*, which refers to abilities in applying mathematics, and to the competences of model building. The importance of technological knowing has been stressed by the applied oriented trend in mathematics education, maintaining that even if students learn mathematics, no guarantee exists that the developed competence is sufficient when it comes to situations of application. More has to be mastered than pure mathematics in order to apply mathematics.[21] This extra competence, I shall call technological competence. More generally, it is the understanding necessary for using a technological tool in pursuing some technological aims. (3) *Reflective knowing*, which refers to the competence in reflecting upon and evaluating the use of mathematics. Reflections have to do with evaluations of the consequences of technological enterprises.[22]

Returning to the example of child benefits, mathematical knowing is exemplified by the competence in performing the basic algorithms of, for instance, multiplication. Technological knowing is represented in the competence of selecting and applying such algorithms to specific problems, i.e., the students have been involved in a technological task when they tried to calculate the child benefits given the procedure for the distribution. Reflections concern the evaluation of the applications of the algorithms, so the students got closer to reflections when they turned from the question "Did we use the algorithm right?" via "Did we us the right algorithm?" to "Is it possible to use an appropriate algorithm in this case?"[23]

Hans Freudenthal has used the notion of reflection, which makes it possible for him to underline mathematics as a human activity. That has changed the focus of mathematics education from "knowledge of" to "coming to know" and to "knowing".[24] It is important to me to preserve this change of focus, but by my use of "reflection" I try to make a new change of focus: Instead of concentrating on how students come to grasp mathematical ideas and how they develop their (own) mathematical understanding, I use "reflection" to designate an understanding *about* mathematics. I do not, in this context, make any sharp distinction between "knowing", which stresses the aspect of process, and "competence", which also makes open the dimension of a non-verbalized ability, i.e., the tacit dimension of knowing.

The distinction between three types of knowing (or competences) not only concerns education but also mathematics seen as part of a general capacity of system development, i.e., as part of a technological force. In fact the distinction is made relevant because of the thesis about the formatting power of mathematics. As an example of mathematical knowing we can take the number theoretical competence which is presupposed in coding theory, developed as a mathematics discipline. We could, however, also look at the competence which makes coding theory a part of weapon systems. That competence includes a competence in coding, not identical with a mathematical competence, and I call it a technological competence. Thirdly, we could reflect upon the use of different forms of weapon technology. What are the consequences of developing weapons with a new power and precision? etc. The competence in making such reflections I call reflective.

The fundamental thesis relating technological and reflective knowing is that technological knowing itself is unable to predict and analyse the results of its own production. Reflections are needed. Technological knowing is born shortsighted. Reflections must be based on a wider horizon of interpretations and pre-understandings. It has to grasp the situation in which technological knowing is at work. Technological and reflective knowing constitute two different types of knowledge, but not two independent types. It may be important to master some technological insight to support reflections. Even if we collect every bit of technological information, we shall not be able to build up reflections from those parts alone. While technological knowing aims at solving a problem, the object for reflections is an evaluation of a suggested technological solution to some (technological) problems.

As part of our culture, structured by technology, a competence in recognizing and interpreting mathematics as a social activity and institution becomes important. And the idea which I have tried to provide with meaning – not to prove – is: *If mathemacy has a role to play in education, similar to but not identical to the role of literacy, then mathemacy must be seen as composed of different competences: a mathematical, a technological and a reflective. And especially: reflective knowing has to be developed to provide mathemacy with a critical dimension.*

If that thesis is acceptable it means that most of the epistemological approaches used in interpreting phenomena in mathematics education are misguiding or at least biases in concentrating on mathematics, ignoring the conditions for the genesis of reflective knowing. The genetic epistemology of Jean Piaget concentrates on the roots of mathematics, the same limitation is found in the different constructivist approaches.[25] The development of a mathematical competence is put into focus. Those approaches all need reinterpretation, criticism and changes to be developed into an epistemology of critical mathematics education.

Reflections on the applications of formal methods is an important element in identifying conditions for social life in a highly technological society. That means that the guiding principles for mathematics education have to be raised to a metalevel, they are no longer to be found in mathematics, nor in any epistemological theory that focuses on the development of mathematical knowledge as such.[26] The only task of mathematics education cannot be the transmission of mathematical concepts to the students. The discussion of the content of mathematics education has also to be guided by the question whether or not it will be able to clarify the actual function of formal methods in today's societies. Could mathematics education introduce students to a discussion of some of the conditions of living in a highly technological society?

The thesis is that mathematics education can become critical in case the competence of mathemacy is developed as a composed competence, including reflective knowing, and that presupposes that the students become involved in an evaluation of technological use of mathematical design. The question then is, wether the project "Family Support in a Micro-Society" illustrates a reasonable educational attempt to establish a situation in which the students get possibilities to develop reflective competence as part of their mathemacy.

"REFLECTIVE KNOWING" IN THE PROJECT

In a mathematical modelling process a transition between different language games occurs. Similar transitions can be found in the project "Family Support in a Micro-Society". The Family Circle constitutes a description of "reality" in natural language. The first transition took place when the data base was established. The students had to go through the descriptions of the families, and, moreover, had to pick up what was relevant from the jumble of information, in order to set up a data base written in a systemic language. This step is similar to that taken in any process of a system development.

Such a transition presupposes interpretations of reality, and at least two types of interpretation took place in creating the students' data base. First, it had to be decided what information was relevant, and that decision depends on the selected criteria for the distribution. For instance, it had to be decided whether the ages of the children were relevant – and what about the age of the parents? And how to install data about a family in which the parents are separated, and the husband lives together with his secretary, while the children most of the time live with their mother, and the father is paying an insufficient amount of money in support? Both the decision about the scope of the relevant information and crystallizing information into "data" presuppose a certain perspective. Data do not mirror any objective reality but represent a condensed perspective. This idea is repeated in the project.

A second transition takes place when a mathematical algorithm of distribution is specified. It also turned out during the project work that making an algorithm for the distribution influenced the students' guidelines for the distribution. They changed the principles of distribution according to what they thought of as manageable implicitly. The two transitions, one from natural language to a systemic language and one from a systemic language to a mathematical and algorithmic language, became the principle transitions in the project.

To illustrate the formatting power of mathematics was an essential aspect of distributing family support. Mathematics freezes an algorithm for what to do, and in that sense the (implicit) mathematical model becomes a guideline for an action. The model manifests norms for distribution. A system is developed. Features of reality can be structured according to formal guidelines, and "reality" becomes the dependent variable. We rearrange "reality" according to mathematics. By means of mathematics we have made available a description of a structure before the structure itself exists, to use a formulation by Davis and Hersh.

Different fundamental questions are connected with formatting. One has to do with the disharmony between what is seen as worthwhile and what looks workable. This disharmony is found in a more general version when solutions to a social problem are pursued by technological means. The conflict may take the following form: We find the best way of solving a problem to be so-and-so, however the only technically feasible solution is so-and-so. During a process of system development a discussion referring to visions and ideals becomes substituted by a discussion referring to technical possibilities. This is a common aspect of the technological culture, and this has to be addressed by a critical activity. The step from a discussion, which contains a normative dimension, to a discussion about what is workable, also takes the form of a step from one type of language games into an other, a step from natural language into a systemic or a mathematical language.

An educational problem is whether or not it is possible to create a situation in which the students come to realize such a conflict. However, during the development of the algorithms for the distribution of support the students got close to problems of that nature: – We cannot do it in that way, it takes too long a time! If we try to take that factor into consideration, we shall never get finished! During the project this conflict was sometimes solved just simply by the students "forgetting" about their original guidelines. Also they wanted to finish their jobs. During a process of system development, visions and ideal actions are modified into manageable calculations. The discussion of norms and values can easily be eliminated by the discussion of technical possibilities, and this limitation is performed as part of the transitions between the different language games involved in a mathematical modelling process. Therefore, the linguistic transitions also mean objectivation and that is a general aspect of mathematical formatting. To become aware of such phenomena must also be a task of a critical mathematics education. These considerations indicate to me that a situation, which makes it possible to establish a discussion essential for making reflections, is found in "Family Support in a Micro-Society".[27]

"Reflection" itself is an open concept, so it may include much more than questions having to do with mathematical modelling. The students made the descriptions of the families: – What to include? Does the family we are going to describe look like my own family? In which family do I prefer to live? The teacher asks us to include some descriptions of poor families. Why does he ask for that? What does being poor really mean? The students were writing small letters to some of the families to explain about the support the family had received: – The families were imaginary, so why to write those letters? The teacher wants us to do so, so anyway let us make some letters. Well, how to explain how we in fact did the calculations? The results of the distributions were written on the blackboard while the teacher read aloud from the descriptions of the families: – Did we really take everything about that family into consideration? Strange, but I seem to have forgotten many things about the different descriptions? Who has written that story? Could it be the teacher, or perhaps that fellow from the university, who has invented the story? There seems to be a great difference between the suggestions from the groups. And how is it possible that the group over there always gives less than the rest of us? They cannot have used all the money. They must have been rather lazy when doing the project. The teacher wants us to discuss the results, but what does he mean? What are we going to discuss? – I am rather sure that we do not receive that much money at home. How much money do we in fact receive?

CRITICAL EDUCATION – A GREATER COMPLEXITY

Is "Family Support in a Micro-Society" an example of critical mathematics education? Or would it maybe be an example, if the students had continued the project and had been involved in discussing the more general question about the use of mathematics and had come to see the difficulties in using mathematics as a principle for design? To answer such questions one needs to know whether the students in fact obtained an awareness useful for making interpretations or not. This does not only presuppose that the project had been carried further, but also that quite different investigations had been carried out. Many aspects, like for instance the nature of the communication between students and teacher, could turn an educational situation into a caricature of any image of a critical mathematics education.

However, let me again emphasise that the project "Family Support in a Micro-Society" does not demonstrate what critical mathematics education has to be. But it illustrates an effort to provide the following expression with educational meaning: Mathemacy can be used for the purpose of empowerment, because it can be a means to organize and reorganize interpretations of social institutions, traditions and proposals for political reforms.

I also have to emphasise that critical mathematics education contains many more aspects than mentioned here. Critical education can refer to the claim of an equal distribution of possibilities for education within a democratic society. And it can imply a concern for the self-confidence of the students which is also included in the general remark, borrowed from Giroux, about literacy (and therefore also about mathemacy) used for the purpose of empowerment. Further, the project "Family Support in a Micro-Society" can be interpreted along different other lines than was here, like for instance the importance of not organising streaming and setting; the project was set up for a mixed ability group. (Even the expression "mixed ability" is suspicious, seen from the perspective of a critical mathematics education.) That means that critical mathematics education cannot be described solely by concentrating on reflective knowing, unless reflections also encompass the educational situation as such.

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NOTES

[1] The example discussed here is investigated in greater detail in *Towards a Philosophy of Critical Mathematics Education* which is going to be published by Kluwer Academic Publishers in Mathematics Education Library.

[2] See for instance Boos-Bavnbek and Pate (1989).

[3] See Kitcher (1984) for a discussion of the mathematical research paradigm.

[4] See also Keitel (1989) and Keitel et al. (1992).

[5] Reprinted in Adorno (1971).

[6] See for instance Paffrath (ed.) (1987).

[7] An excellent English introduction to Critical Theory is found in Held (1980). See also Benhabib (1986), Connerton (1980), Jay (1973) and Kellner (1989).

[8] See for instance Negt (1965) and Illeris (1974).

[9] The German word *Fachkritik* designates the activity of "going behind" the curriculum and ask for the logical, sociological and political assumptions which constitute the subject matter as such.

[10] For an English introduction see Young (1989).
[11] See for instance the discussion of the social reconstructionist movement, closely related to the work of John Dewey, in Giroux (1989). Also of interest is Livingstone (ed.) (1987).
[12] See Freire (1972, 1974).
[13] See for instance the discussion of literacy in Giroux (1989).
[14] The term "mathemacy" has been used by Ubiratan D'Ambrosio in different of his works on ethnomathematics (1980, 1981, 1985).
[15] See also Abraham and Bibby (1988), Borba (1990). Frankenstein (1987, 1989), Münzinger (1977), Mellin-Olsen (1987), Niss (1977, 1983), Skovsmose (1980, 1981a,b, 1984), and Volk (1975, 1979).
[16] See also Skovsmose (1985, 1988, 1990a, 1992).
[17] Child benefit in Denmark is a regular payment from the Government and the Districts to families with children.
[18] One possibility had been to use the children's own families as source of the description, but that possibility could cause problems and uncertainty, so we prefered to let the Micro-Society be imaginary.
[19] It became obvious that the facility of the data base influenced the students' way of handling the distribution. It became much easier to think in terms of steps instead of proportionality.
[20] How much mathematics is in fact included in such a task of making a distribution of a certain amount of money according to some selected principles? The Family Circle has been presented by Poul Hjarnaa and Andreas Reinholt to groups of mathematics students from Aalborg Teacher Training College. The groups worked intensively with the problem of making an algorithm for the distributions, and one of the groups came up with the following question: Does the following formula

$$F_i = \left(\frac{y_i}{\left(\frac{\left(\frac{x_i}{y_i}\right)}{\sum_{j=1}^{24} x_j} \right)} \Big/ \sum_{k=1}^{24} \left(\frac{y_k}{\left(\frac{\left(\frac{x_k}{y_k}\right)}{\sum_{j=1}^{24} x_j} \right)} \right) \right) \cdot 240,000$$

where x_i designates the income and y_i the number of children of the family i, describe the support F_i which the family should receive? Could the formula be simplified?
[21] See Niss (1985, 1989). Those papers also explicate a critique of a narrow application oriented mathematics education.
[22] Naturally, by this specification, I do not maintain that no other types of competences are at stage in a mathematics education. I have just mentioned three types related to the subject matter in question. Another competence, possessed by the students, has to do with interpreting the school situations.
[23] See also Skovsmose (1989a,b, 1990b, 1992) for a discussion of reflective knowledge. Also of importance, but with a different perspective is Gagatsis and Patronis (1990).
[24] See Chapter 2, Section 2.3.5. in Freudenthal (1991). Here Freudenthal gives a short overview of how he sees "reflections" in his work.
[25] See Beth and Piaget (1966) and Piaget (1970).
[26] See also Davis (1989). A similar conclusion has also been pointed out by Mogens Niss:

It is of democratic importance, to the individual as well as to society at large, that any citizen is provided with instruments for understanding the role of mathematics. Anyone not in position of such instruments becomes "victim" of social processes in which mathematics is a component. So, the purpose of mathematical education should be to enable students to realize, understand, judge, utilize and also to perform the application of mathematics in society, in particular in situations which are of significance to their private, social and professional life.

(Niss, 1983, p. 248). Of special importance is also the discussion of the public educator programme in Ernest (1991).

[27] These comments have been somewhat negative. The terms used have been: simplification, making things manageable, etc. As set up in this section, mathematical formatting primarily is seen as problematic, and reflections become directed towards formulating limitations and difficulties. The mode has partly been the gloomy one borrowed from Ellul (1964): We are captured in technology, and we come to act by means of technology according to standards set out by technology. Naturally, it may be possible to find routes the other way around. By the use of mathematics some possible solutions, not possible to imagine without mathematics, can be found. Modelling has to do with a transition between language games, and the grammar of mathematics also provides possibilities not found in natural language.

It is important not to think of the transitions from verbal criteria to mathematics as steps which always lead to undesirable simplifications and limitations. During a system development something may be lost, but we could also gain something different. If the distribution of family support should take place without any sort of algorithms, then the distribution may be subjected to arbitrary guidelines. To develop a mathematical algorithm also means a way of discussing and obtaining universality. This dimension has also to be reflected upon.

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