In Praise of Amateurs

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One of the most appealing aspects of Martin Gardner's column "Mathematical Games" is its presentation of mathematical problems designed to intrigue amateurs and encourage their personal efforts at solution. By his own insistence, Gardner is an amateur mathematician and gives no special deference to formal mathematical education—his column pays tribute to the efforts of the mathematical "great" and the mathematically unknown, names often appearing side by side with no titles to distinguish one from the other. Amateurs are his most avid followers and enjoy the challenge of matching their wits against others in solving problems. Amazingly, their lack of formal mathematical education is often an advantage rather than a hindrance and their ingenious solutions to problems sometimes top the efforts of the professionals.

A striking case in point occurred as a result of Gardner's July 1975 column "On Tessellating the Plane with Convex Polygon Tiles". Challenged by the column, Richard James decided to try his own hand at solution and his obvious approach (altering the familiar) produced a solution that was overlooked in a formal mathematical scheme. The subsequent report of James's discovery aroused intense curiosity in Marjorie Rice, providing her with staying power for a thorough and methodical search (carried out mostly at her kitchen counter), which ultimately yielded a wealth of new results. The invitation to write this article gives me the chance to relate some details of the events that one Gardner column set

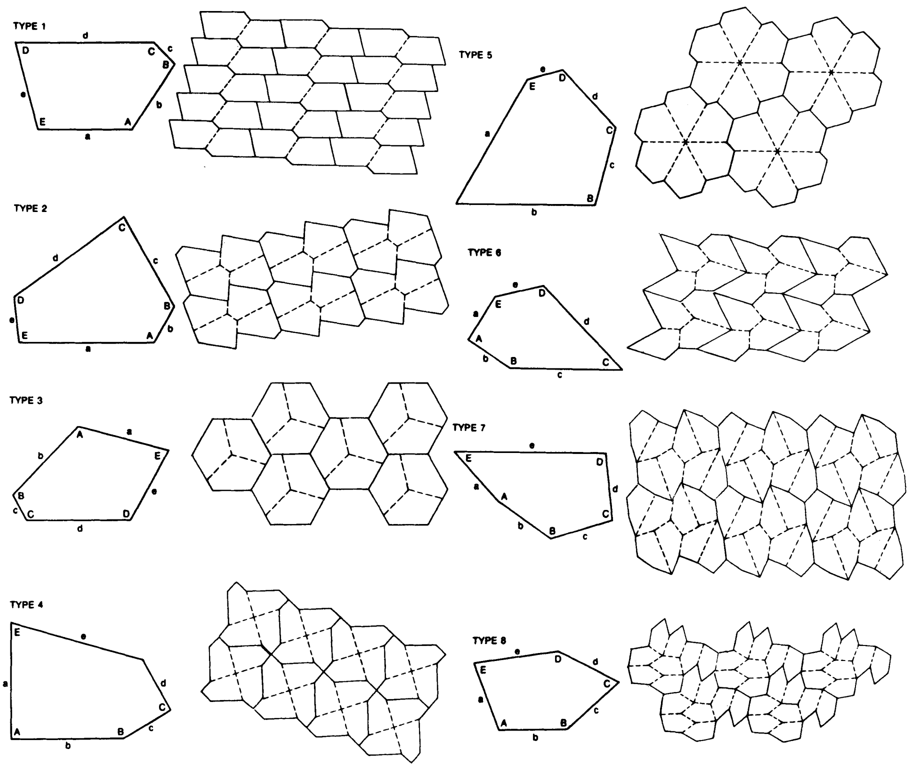
D. A. Klarner (ed.), *The Mathematical Gardner*

in motion—events that still continue to ripple. It is a tribute to the thousands of amateurs who have made Gardner's column such a success.

Tiling problems have been a favorite subject of Gardner's over the years. More than a dozen columns in the last twenty years have been devoted in large measure to this subject. How to fit pieces together snugly to fill a desired space seems to be one of our earliest childhood pastimes. Even as adults we continue to be engaged by these problems, either for pleasure or out of necessity, with tiles, bricks and the like. One of the most basic tiling questions to ask is "What shape tile will fill the plane with its replicas without gaps or overlaps?" Many obvious shapes come to mind—no mathematician is needed to supply a lengthy list of examples. Some of the beautifully shaped tiles found in ancient mosaics around the world attest to the imagination of decorative artists in solving the problem. The most general answer to our question is not known. In order to get partial answers, conditions on the tiles are specified and then these specialized questions become problems whose solution is sought. What if the tiles are composed of stuck-together squares (polyominoes)? Or equilateral triangles (polyiamonds)? Or regular hexagons (polyhexes)?

Gardner's July 1975 article discussed the problem in which the tiles were convex polygons. "What convex polygons will tile? Explicitly describe conditions on a convex polygon to insure that it tiles." Here, Gardner had chosen a topic which had been worked on by many mathematicians over a 50-year period and it had been announced that the problem was completely solved. It is easily discovered that any triangle or any quadrilateral can tile, but that convex polygons of five or more sides do not always tile. For instance, regular pentagons do not tile but any pentagon having a pair of parallel sides will tile. Regular hexagons do tile but many other hexagons do not. Convex polygons having seven or more sides cannot tile. This last statement declared by Gardner as "not hard to show", is best described as a mathematical "folk theorem". Thus, everyone quotes it, but no one can seem to cite a complete and accurate proof. Fortunately, a recent article by Ivan Niven, in the December 1978 *American Mathematical Monthly*, fills this gap in the mathematical literature and presents a thorough as well as convincing proof.

Gardner based his article on a 1968 paper by R. B. Kershner, which surveyed the answers to the question, "What convex polygons will tile?" The complete answer for hexagons (3 types) and partial answer for pentagons (5 types) had been found by K. Reinhardt in 1918, and extensive investigations by Kershner had yielded 3 more types of pentagons which tile. Kershner felt the list was now complete (Figure 1). His fascination with the problem was described in a letter quoted by Gardner: "For reasons that I would have difficulty explaining, I have been intrigued by this problem for some thirty-five years. Every five or ten years I have made some kind of attempt to solve the problem. Some two years ago I finally discovered a method of classifying the possibilities for pentagons in a more



1. $A + B + C = 360°$.
2. $A + B + D = 360°$, and $a = d$.
3. $A = C = D = 120°$, and $a = b$, $d = c + e$
4. $A = C = 90°$, and $a = b$, $c = d$.
5. $A = 60°$, $C = 120°$, and $a = b$, $c = d$.
6. $A + B + D = 360°$. $A = 2C$, and $a = b = e$, $c = d$.
7. $2B + C = 2D + A = 360°$, and $a = b = c = d$.
8. $2A + B = 2D + C = 360°$, and $a = b = c = d$.

FIGURE 1

The eight types of convex pentagons which tile, reported by Martin Gardner in July, 1975.

convenient way than Reinhardt's. ... The result of this investigation was the discovery that there were just three additional types of pentagon ... that can pave the plane. These pavings are totally surprising. The discovery of their existence is a source of considerable gratification." Kershner's fascination with the problem would prove to be contagious. Gardner's exposition removed the subject from dusty mathematical journals (where it had been unquestioned for many years) and placed it in the hands of a wide readership, including many amateur puzzle enthusiasts. It is here that our story begins.

When Richard James III saw Gardner's article, he read only the first part; he decided to test his puzzle-solving skills before reading the remainder of the article. He wrote (in a letter kindly supplied by H. S. M. Coxeter), "Before reading Mr. Gardner's description of Mr. Kershner's eight tessellating pentagons, I set about to try it for myself. The first thing that came to mind was taking ... octagons (with squares filling the holes) and adapting them so that pentagons replace the squares (thus moving the octagons out of a lattice into "parallel" strips). The octagons neatly split into four pentagons. The description was $A = B = E = 90°$, $C = D = 135°$, and $a = b = 2c = 2e$. The tessellation was interesting but the pentagon was dull. Rotating the cross in the octagon (and making other adjustments as needed) produced [the pentagon and tiling shown in Figure 2]." James sent his discovery to Martin Gardner with the inquiry: "Do you agree that Kershner missed this one?" The exciting news was immediately communicated by Gardner to Kershner and to a few other mathematicians. Kershner's good-humored response and the James tessellation were reported to readers in the December 1975 "Mathematical Games" column. An amateur had, with one new example, shown that the list of tessellating pentagons was *not* complete. Were there still others to be discovered?

Perhaps an aside is in order here; a glimpse at the "mathematical grapevine" maintained by Gardner (which I would like to denote MG^2, for *Martin Gardner's mathematical grapevine*—shown in Figure 3). MG^2 is kept humming year round by a steady flow of correspondence, telephone calls, and personal conversations. When researching or writing a column, Gardner contacts experts with knowledge and/or the latest information on the subject; he seeks their comments on the accuracy of his manuscript before submitting it for publication. Conversely, when Gardner receives correspondence on any problem, before filing it away for future reference, he sends copies to those he knows are actively interested. In this way, the latest news is made available to those interested; its accuracy is checked and comments are returned to Gardner; and most importantly, those working on a common problem are put in touch with each other by Gardner. H. S. M. Coxeter was one of those who received a copy of James's discovery from Gardner; it was from Coxeter that I learned of the discovery. I couldn't resist trying my own hand at the problem—it wasn't long before I had a general description of a class of tessellating pentagons which generalized the single example James had sent to Gardner. Following Kershner's method of description, the general pentagon of James's type is one which satisfies the equations: $A = 90°$, $E = 180° - B$, $D = 90° + B/2$, $C = 180° - B/2$, $a = b = c + e$. After my communication of this to Gardner and Coxeter, I found myself the recipient of copies of Gardner's correspondence on "the pentagon problem" and soon was in direct communication with others actively working on the problem. If my bulging files on this problem can be taken as an indicator of the correspondence generated by one

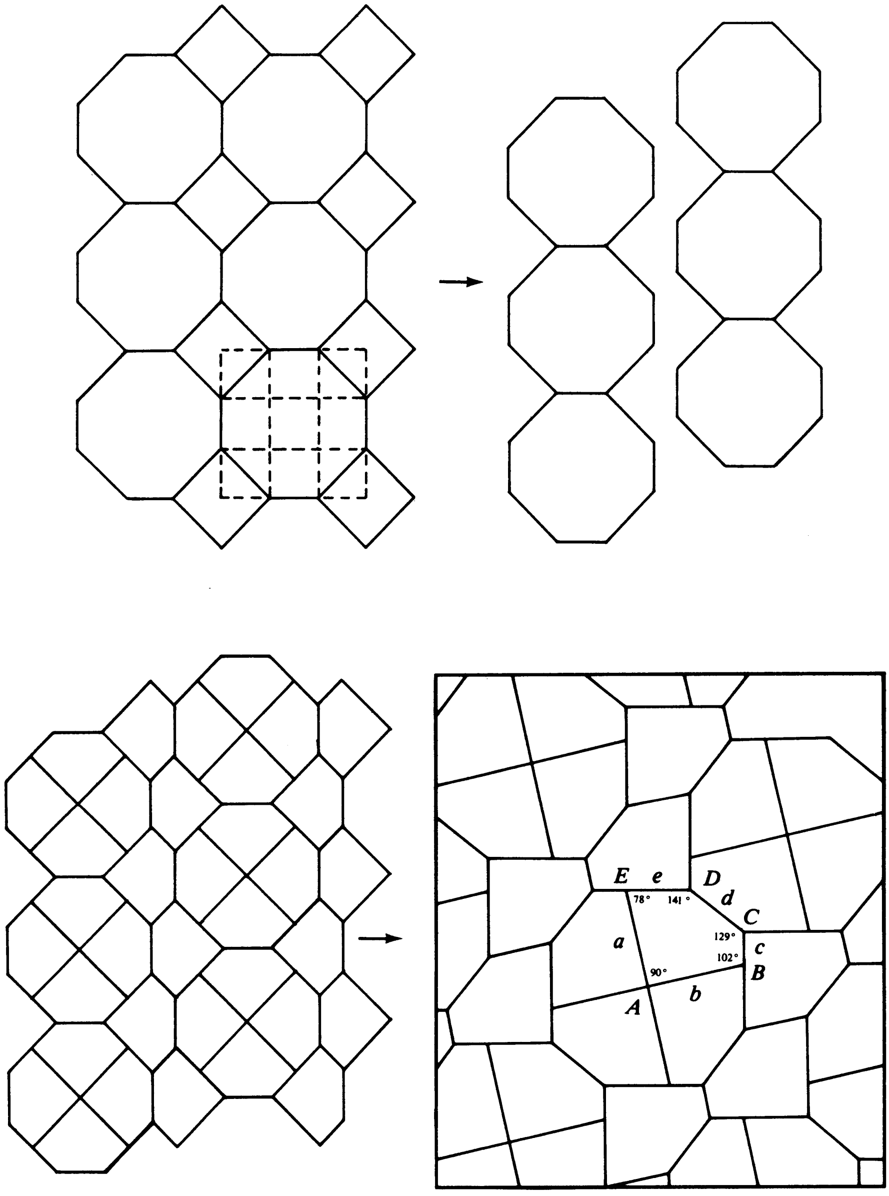


FIGURE 2

HOW RICHARD JAMES DISCOVERED A NEW PENTAGON THAT TILES

The familiar tiling by octagons and squares (with underlying square grid in dotted outline) is shifted to see if pentagons might replace the squares. A successful new tiling is produced. Further alteration of the octagons and pentagons produces an example of a previously undiscovered family of pentagons which tile.

FIGURE 3
MG^2

Gardner column, then his files surely must fill several rooms!

When the December 1975 issue of *Scientific American* was delivered, another avid Gardner fan in California turned immediately to the "Mathematical Games" column. Marjorie Rice, a San Diego housewife and mother of five, was usually the first one in the household to read her son's magazine. She had been intrigued by the July article on tiling by pentagons and had "thought how wonderful it must have been [for Kershner] to discover the new types of pentagon tiles." Now, reading of James's newly discovered pentagon tile, her interest was strongly aroused and she set out to see if she might find still other new pentagons which tile. "I thought I would like to understand these fascinating patterns better and see if I could find still another type. It was like a delightful new puzzle to me and I considered how I could best go about this." Her search began quite differently from that of James and soon became a full-scale assault on the problem, extending over a period of two years.

Marjorie Rice had no formal education in mathematics beyond a General Mathematics course required for graduation from high school in 1939. Thus, as she faced the challenge of finding new pentagonal tiles she not only worked out her own method of attack, but also invented her own notation as well. Her first step was to catalogue all of the information available in the two Gardner columns on tiling pentagons (Figure 4). By doing this, she hoped to discover any common relationships satisfied by the pentagons and their tilings. "To begin, I needed to visualize the 9 types to

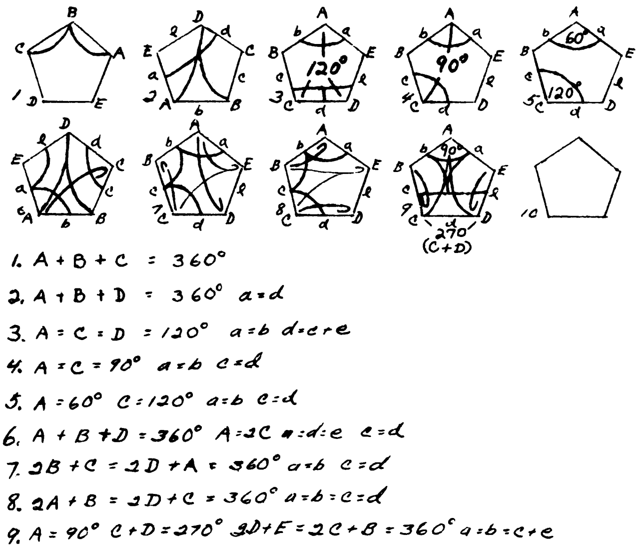


FIGURE 4

Marjorie Rice's codification of information on the tiling pentagons of types 1 through 8 and James's discovery.

see how they differed from one another. I listed the formulas [the equations on sides and angles] on a 3 × 5 card and drew 10 pentagons on another card. I drew lines in color within the pentagons to show the information in the formulas, red for 360° combinations of 3 angles, blue for edges of the same length, black for 360° combinations of 4 angles, greeen for other information." "Now I could see that in [types] 7 and 8 each vertex had been touched twice by a red or black line (the hooks counting for two times) and that if I had drawn a line on [types] 1 or 2 between angles totaling 180°, each vertex would be touched once by a line. Continuing to [type] 3, I used the symbol ⩛ to indicate 3 identical angles would come together, this for 3 of the vertices. Three straight lines were needed then between [angles] *B* and *E*. Every corner was touched three times. I saw that for every pattern each vertex of the pentagon must be touched by a line or symbol the same number of times."

This observation that each vertex of a tiling pentagon must be used the same number of times in a tiling was the key to Marjorie's investigation which followed. Her symbolic notation of information was further refined to a form which suppressed all but essential information (Figure 5). "Here I used [symbolic] pentagons in a form easier to draw." Lines connected corners of the pentagon which would come together at a vertex of the tiling by the pentagon. "This notation is the key to all my further work. By labeling the corners (it didn't matter where I started), I could then develop a sort of signature of letter combinations for every diagram and develop these with little sketches."

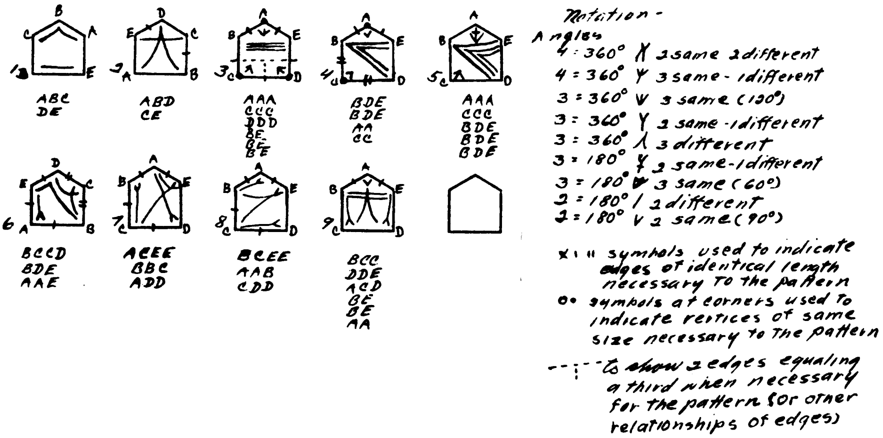


FIGURE 5

The pictorial notation developed by Marjorie Rice (here recording information on the 9 types of pentagons known to tile) which was the key to all her work.

In her own way, Marjorie had discovered a way to manage an enormous amount of information that would emerge in a thorough combinatorial search of possible combinations of angles (and sides) which yield tiling pentagons. Mathematicians use symbols for objectivity, conciseness and clarity—and good notation must be simultaneously suggestive and definitive as well. Marjorie's pictorial notation looks like hieroglyphics yet it records the possible combinations of angles with a simplicity not possible with more conventional mathematical notation. The notation eliminates completely the need to worry about repetition of cases caused by assigning different letters to the angles of a pentagon.

Beginning with two copies of a single pentagon stuck together, Marjorie considered how further copies of the pentagon could be added to these to create a tiling of the plane. The information on how corners of the pentagon met at a vertex of the tiling and how sides touched were all kept track of by using her symbolic notation. If it became clear that a certain combination was impossible (a tiling would not result), this case was eliminated; when a tiling seemed possible she sketched an actual example of such a pentagon and its tiling.

In order to make calculations quickly and test new pentagons to see whether or not their angles might add up so as to create vertices of a new pentagonal tiling, Marjorie arbitrarily divided 360° into units of 18°, marking the divisions on a small protractor. Then using these units, an angle of 36° was represented as 2, an angle of 108° as 6, and so on. "When constructing a pentagon for trial I usually began with the 2 angles that equal 180° numbered as 4 (72°) and 6 (108°) and adjusted them later on as

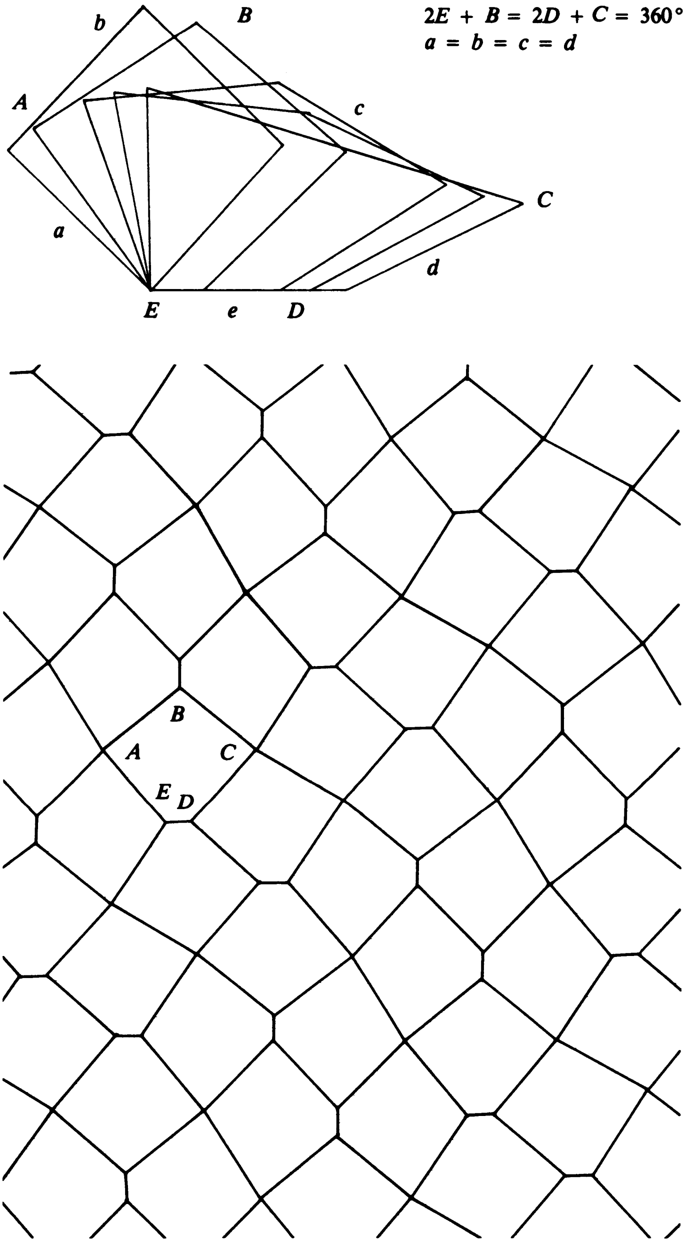


FIGURE 6

The first discovery (February 1976) of a new type of tiling pentagon by Marjorie Rice. The range of shapes that the pentagon can assume is shown along with a tiling by one representative of this type. A tiling by a representative of this type whose sides are in golden ratio is the underlying grid for an Escher-like design of bees in clover, designed by Mrs. Rice. (See Figure 16A.)

required. This was the busy Christmas [1975] season which took much of my time but I got back to the problem whenever I could and began drawing little diagrams on my kitchen counter when no one was there, covering them up quickly if someone came by, for I didn't wish to have to explain what I was doing to anyone. Soon I realized that many interesting patterns were possible but did not pursue them further, for I was searching for a new type and a few weeks later, I found it."

In mid-February, 1976, Marjorie sent her discovery to Gardner with a sketch of the range of shapes which the pentagon could assume and tilings by two different representatives of this new type of pentagon tile (Figure 6). She wrote, "Here is a pentagonal tile that I believe really is different from any you had listed though similar to type 7 and 8. One of the enclosed examples in which the two sizes of line are in golden proportion makes a very pleasing arrangement, I think." Again, Gardner dispatched the discovery to several interested parties via MG^2 including Kershner and myself. It was verified to be indeed a new addition to the list of tiling pentagons; Kershner wrote to Marjorie to ask how she discovered it, and acknowledged that he had erroneously eliminated this as a possible type in his search. As happens with a great deal of reader correspondence, the discovery was not reported in Gardner's column but filed away for future reference. (Other items had also been received by Gardner in response to the James tessellation—at least one quilt and a beautifully woven rug were inspired by the design. See color plate V.)

When I examined the material that Marjorie had sent to Gardner and compared it with Kershner's types 7 and 8, it appeared that these three types of tiling pentagons (I named hers type 9) might all be examples in a still larger class of tiling pentagons. I made the following conjecture and sent it to Gardner: "Any pentagon having four equal sides and containing four different angles $P, Q, R, S,$ such that $2P + Q = 360°$ and $2R + S = 360°$, tiles the plane." In less than two weeks I received a letter from Marjorie in which she showed that she had considered the conjecture and proved it false. "As the symbols below show, there are only 8 possibilities—[8 ways in which the corners of such a pentagon can come together so that only the equations $2P + Q = 360°$ and $2R + S = 360°$ are used]. Four of them *1, 5, 6,* and *8* can tile the plane, the other four seem to be impossible for the reasons illustrated. *6* seems only to work when two adjacent angles equal 180° thus making it a type 1. It does however give two interesting ways of assembling a special type kind of type 1 as shown on the enclosed sheet." (Figure 7).

This was my first correspondence from Marjorie and my first encounter with her notation and method of checking possibilities by construction. It was so far from the conventional ways that mathematicians use that I puzzled over the diagrams trying to figure out what she was saying and how these proved anything. Her "reasons" that some pentagons considered would not tile were little sketches, not algebraic or geometric

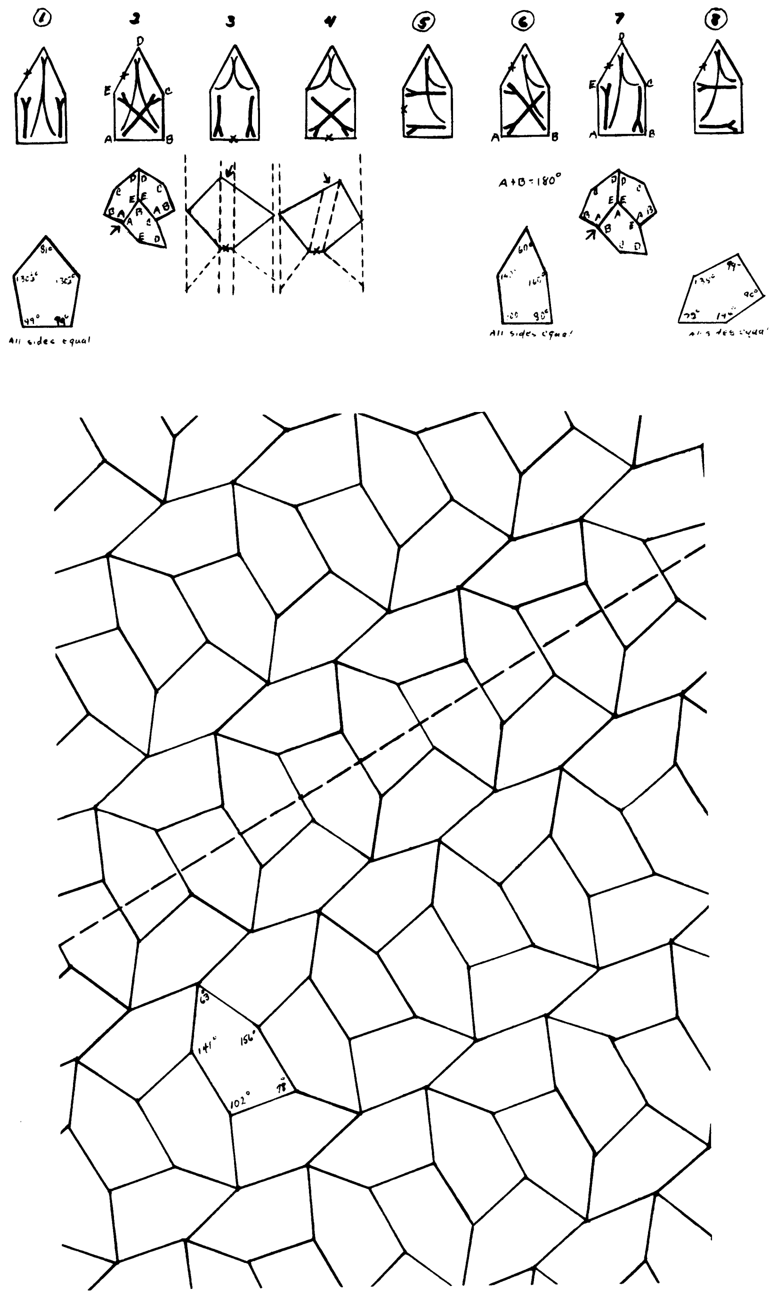


FIGURE 7

The pictorial "proof" that Schatlschneider's conjecture was false. Case *6*, having 4 equal sides and 2 adjacent angles whose sum is 180°, assembles into blocks in 2 distinct ways (note the change in the tiling across the dashed line).

arguments that mathematicians require for proof. Probably because it seemed so obvious to her, she had not enclosed any explanation of her pictorial notation. In her notation, the heavy chicken tracks $\succ$— represent the equations $2P+Q=360°$, $2R+S=360°$ which would bring together these corners at a vertex of the tiling, and the lighter curves connecting corners of a pentagon represent the remaining equation on how corners must come together at a vertex of the tiling. Recall that the sum of the interior angles of a pentagon is 540° so that this fact, together with the two equations involving P, Q, R, S imply that $Q+S+2T=360°$, where T is the 5th angle of the pentagon. She assumed no further equations were necessarily satisfied by angles of a pentagon and so a tiling by that pentagon could use only the angle relationships represented symbolically. Thus her cases *2* and *7* were eliminated because (as her arrow indicates) another equation on angles would have to be added if the pentagons were to tile in this way. Her cases *3* and *4* are impossible to construct—her sketches tell this pictorially. It was not hard for me to verify algebraically that no pentagon could satisfy the relationships on angles shown in these cases. Marjorie had constructed several examples of her case *6* and always 2 adjacent angles seemed to add up to 180°. But this observation did not constitute a mathematical proof. In trying to prove her observation, I found that the assumed angle relationships did not force this fact. It was Kershner who later supplied an elegant proof, showing the usefulness of his generalized laws of sines and cosines. This provided a very striking illustration of an amateur's intuition and observation using elementary tools leading to a correct conclusion, but the necessity of more sophisticated, mathematical means and a trained mathematical mind to provide irrefutable proof.

The four cases which remained did indeed tile and were already known—her cases *1* and *8* were Kershner's types 8 and 7 respectively. Her case *5* was her new discovery (type 9) and case *6* was type 1. Thus the conjecture was completely disposed of.

Now that Marjorie had established direct contact with Kershner and with me, MG^2 was no longer an intermediary; however, Gardner was kept informed of the continuing work on the problem. No doubt Marjorie was encouraged by the praise for her work received in correspondence, but it was the problem itself which continued to entice her. Although she was busy with family events she kept returning to the puzzle, drawing tilings and considering and reconsidering possibilities in whatever snatches of time she could find. The problem was like a partially finished jigsaw puzzle laid out in a spare room—worked on intently for a while until a small satisfaction is achieved, then abandoned. It is not forgotten, however, and lures you back again and again to tempt you to add a few more pieces and see a little more of the pieced-together scene.

I asked Marjorie to write me all she had done on the problem and keep me informed of any new results since I had been asked to write an

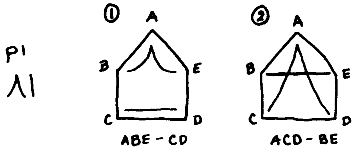


FIGURE 8

Symbolic representation of pentagons in which each angle is "used" once in a vertex of the tiling.

article on the pentagonal tiling problem for *Mathematics Magazine*. In March, 1976, I received her codified analysis of how she had considered ways in which a pentagon could tile. The diagram showed groups of pentagons considered—each group corresponded to a set of angle relationships satisfied by a pentagon. These angle relationships (and only these) were to be reflected in an associated tiling of the plane by the pentagon. How the angles came together then forced any conditions on the sides of a tiling pentagon. The first group, called "*p*1" (for pentagon-1), considered pentagons in which each angle was "used" once in a vertex of the tiling. Thus three different angles "came together" to sum to 360° and the two remaining angles "came together" to sum to 180°. Only 2 pentagons, types 1 and 2, were in this category (Figure 8). The next 12 groups (listed 1 through 12 under a heading "*p*2") considered pentagons in which each angle was used twice if different vertices of the tiling were listed. Group 12 in this listing is the collection of pentagons which Marjorie had written to me about earlier, in response to my conjecture. On this codified listing (Figure 9), Marjorie explained "There are 3 tests, the first is whether the group itself is possible (5 and 6 are not). Then sketches are made of combinations indicated by each member of a group to see if the proper angles will come together. If they combine successfully, it will be obvious which lines [sides of pentagons] must be equal lengths. The last test is whether it can be translated into specific angles—if it can, it will tile successfully." Marjorie had written "no" next to each symbolic pentagon in the list which would not tile in the way specified; next to those that would tile, she put the type number from Kershner's list. In addition, for each pentagon that tiled she produced an illustrative tiling showing that it could tile in the way specified (Figures 9a, 9b). In her listing, types 1, 2, 4, 6, 7, 8, 9 had occurred and she had 26 different tilings. Several of these tilings were new. It is not surprising that types 3, 5, and James's tile (which I call type 10) did not occur on this list because if all of the angle relationships which are satisfied at the vertices of tilings by these types are given, then each angle of a pentagon tile is used three times.

A few weeks after receiving this information, Marjorie sent an even more extensive list. "Regarding the 2-pentagon patterns I sent you earlier—I have found on rechecking that I had missed quite a few—have gone over them much more carefully and here is the revised list with examples." Her revised list showed the same 12 groups but many more cases considered. Now she had found 35 pentagons with angle relationships that led to plane tilings. Some angle combinations yielded two or more distinct tilings; thus there were a total of 45 sketched tilings accompanying this listing. Although no new types of tiles were discovered, the range of tilings was greatly expanded. Her letter indicated that she was already at work on the "*p*3" groups of pentagons—those whose angle relationships would use each angle of a pentagon three times. "The majority are quickly

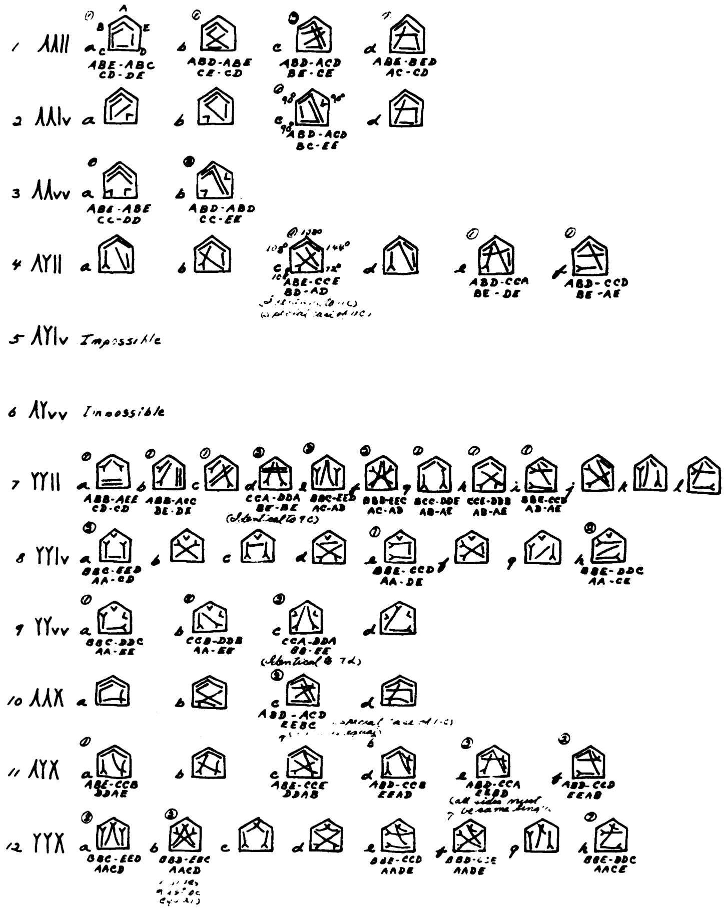


FIGURE 9

The 12 groups of pentagons considered which might form "2-pentagon" patterns. Each angle of such a pentagon is "used" twice in the list of angle relationships occuring at the vertices of the tiling. Sketches of tilings are shown in Figures 9A and 9B for each successful combination of angles. This listing and collection of tilings is the second one made by Marjorie Rice.

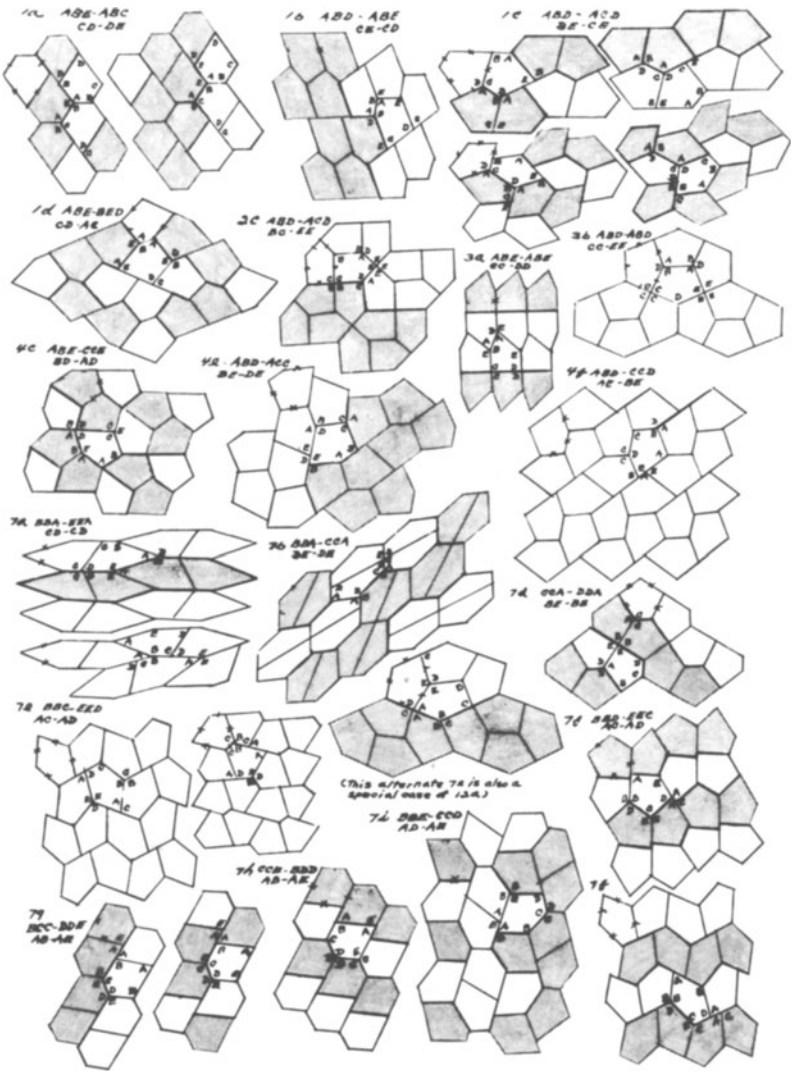


FIGURE 9A

seen to be impossible, so it doesn't look like such a formidable job to go through those remaining. ... Among them are types 3 and 5 and Mr. James's new one [type 10]."

In October, 1976, I received another bulging envelope from Marjorie. She had made a new listing of all of the pentagon tilings she had discovered thus far—58 in all. She had reorganized her listing; this time she had arranged the pentagons (and associated tilings) into 12 classes, each class corresponding to which sides of the pentagon must be equal. Six pages of illustrative tilings demonstrated the thoroughness of her work.

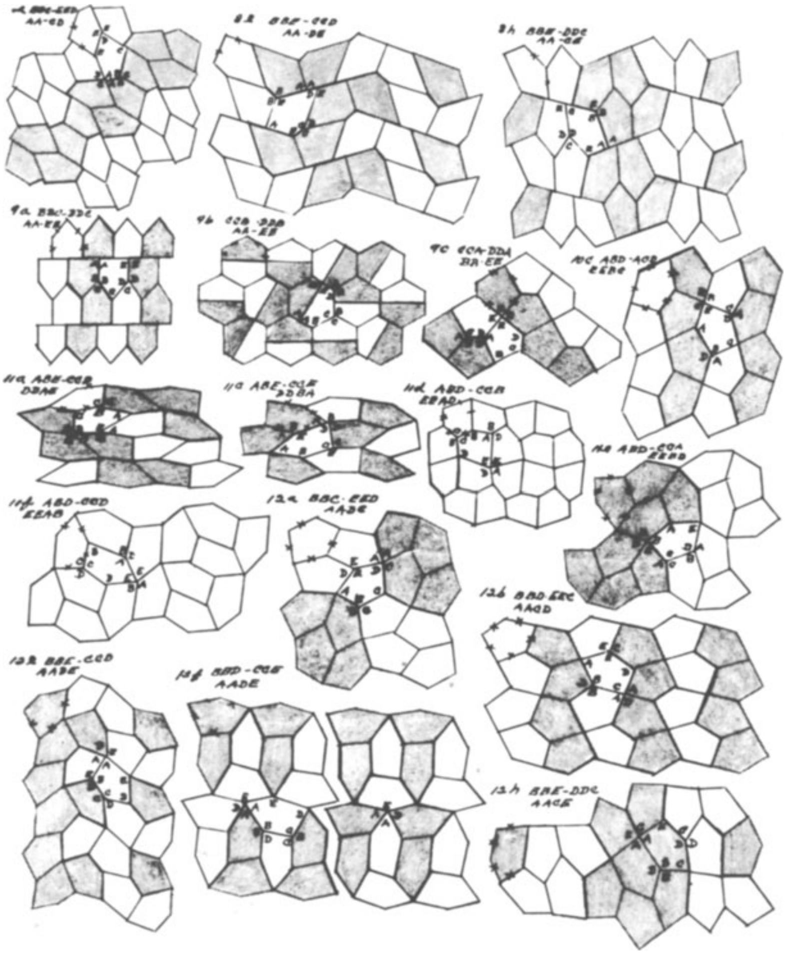


FIGURE 9B

Every one of the ten types of tiling pentagons occurred, and there were many new tilings. She closed her letter "this is as far as I can go with my limited knowledge, so I am through looking. Perhaps there are still others I have not come across."

In mid-November, I sent Marjorie a preprint of a paper by Branko Grünbaum and Geoffrey Shephard which showed the 24 tile-transitive tilings by pentagons of types 1 through 5 and also the first draft of my article on the pentagon problem for publication in *Mathematics Magazine*. This was an expanded version of a talk I had given at a Conference on Recreational Mathematics at Miami University in Oxford, Ohio. John H. Conway had been at that conference and showed great interest in the problem and the contributions made by James and Rice. He admitted that he had once set out to find all tiling pentagons but had abandoned the problem when it became too time-consuming. At the end of my article I raised some natural questions: "Is the list of pentagons which can tile in an edge-to-edge manner complete?" "Can we find the complete list of all equilateral pentagons which tile?"

Marjorie couldn't ignore the questions. I received another letter in December, 1976. "Had thought to spend no more time on pentagons but they weren't so easy to lay aside." This time she had tackled the questions raised in my article. In reply to the last question, she had sketched all of her tilings by equilateral pentagons and the angle relationships forced by the tilings. She had also tackled the first question. In the article, I had explained "block-transitive" tiling by pentagons, noting that all of the recent discoveries—Kershner's, James's, and hers were pentagons which could not tile isohedrally, but for which a block of two or three stuck-together pentagons was necessary to generate the tiling. This notion was new to Marjorie and she reexamined all of her tilings which began with two pentagons stuck together and noticed that "most of them consist of 4 tiles forming 2 hexagons which tile in one of 6 ways." Focusing in on one possible dissection of such a block into four congruent pentagons had led her to discover several new edge-to-edge tilings. Two weeks later (December 27, 1976—the Christmas season seemed to be her most creative time!), came the exciting news: "I have been working with this idea further and have some new patterns, and to my surprise and delight—2 new types, closely related." Indeed, she had discovered types 11 and 12, and the accompanying tilings were very striking (Figure 10).

The discovery of the new types had been an unexpected bonus in her methodical analysis of 2-block transitive designs. She had found that the double hexagons could be dissected into four congruent pentagons in nine distinct ways (Figure 11) and these dissections together with the variety of

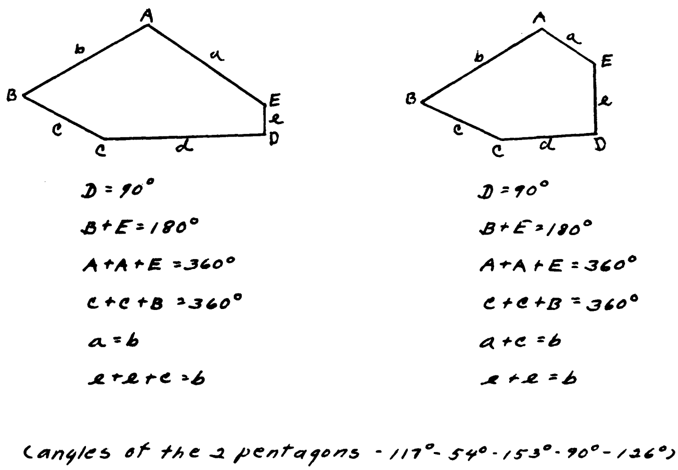


FIGURE 10

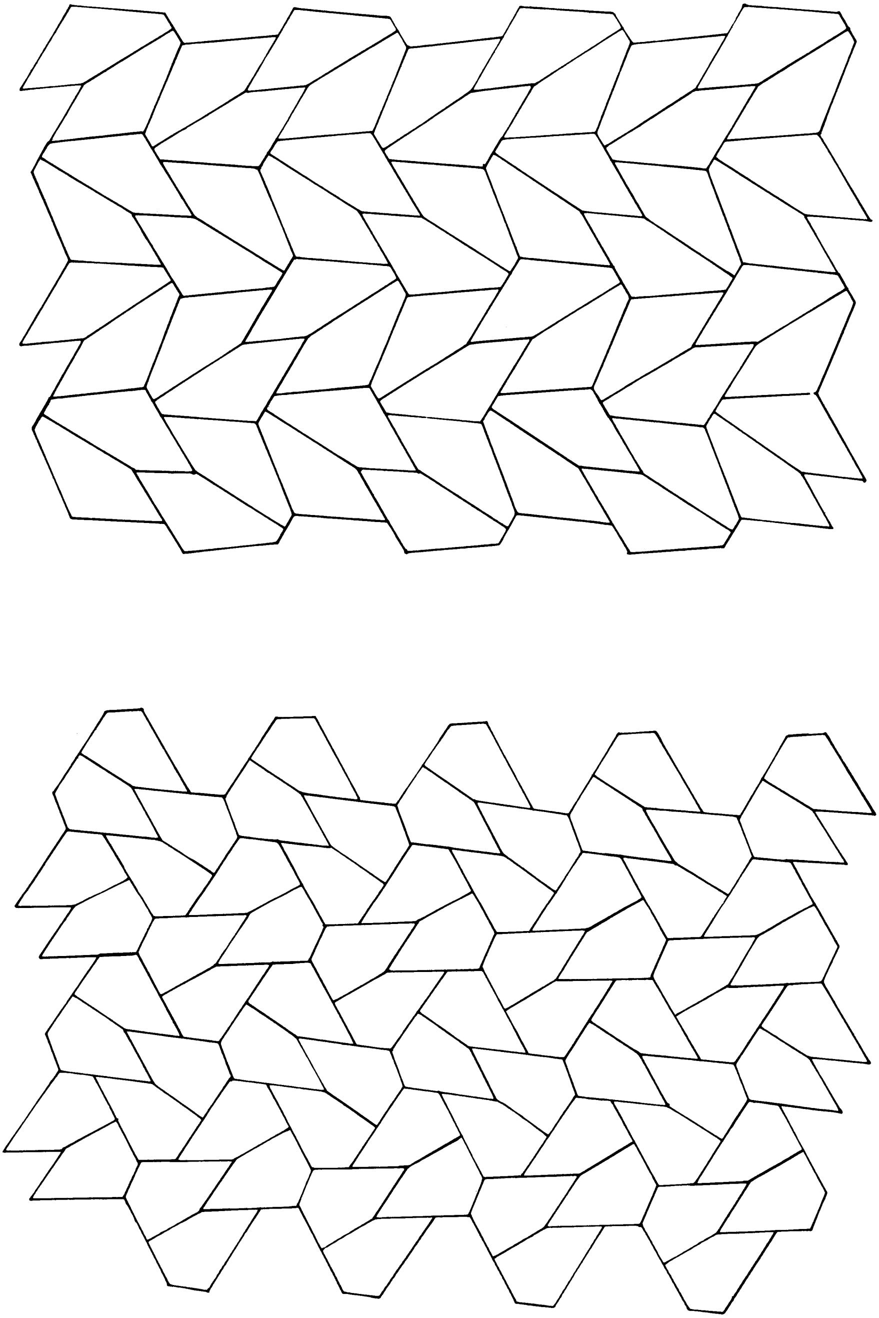


FIGURE 10 (cont'd)

Tiling pentagons of types 11 and 12 discovered by Marjorie Rice in December, 1976.

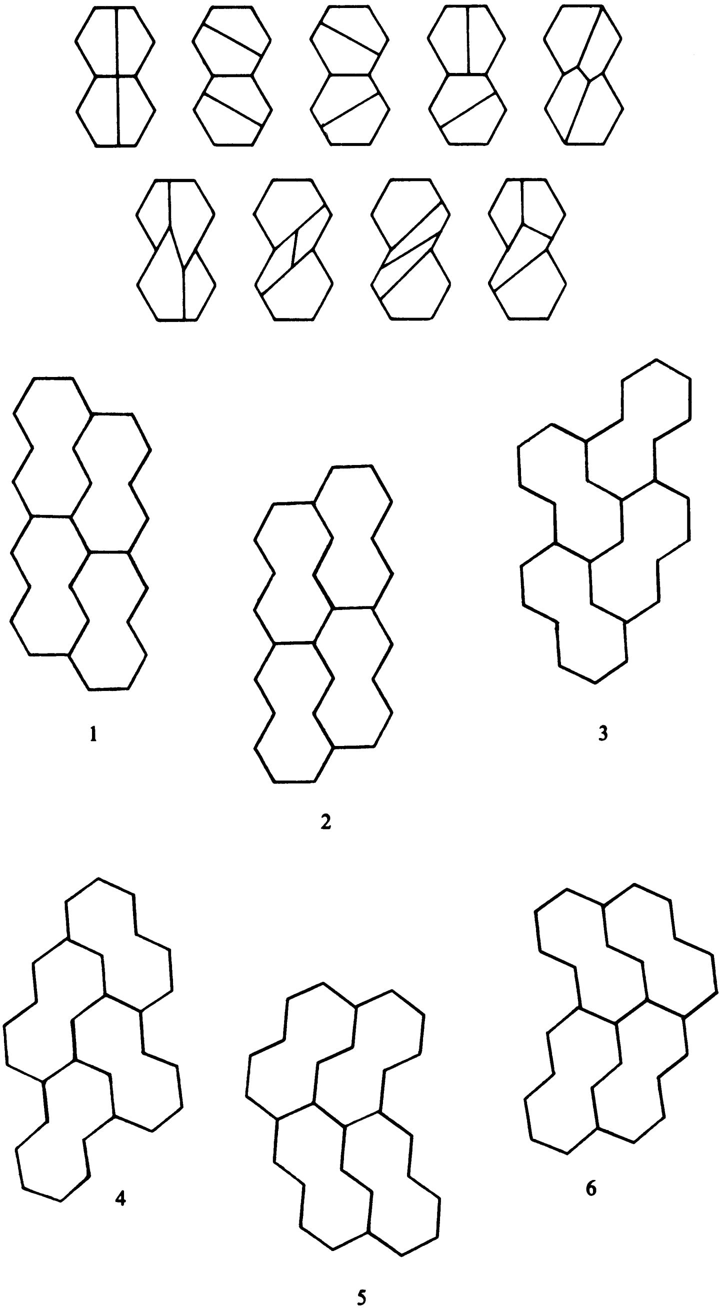


FIGURE 11

Consideration of how "double hexagon" blocks (represented here symbolically) could be dissected into four congruent pentagons and the ways in which such double hexagon blocks can tile, produced a wealth of new tilings by pentagons.

tilings by their blocks, led to over 50 different tilings by pentagons (the tilings were "2-block transitive"). Through the spring she continued to pursue the idea and found several dissections leading to 3-block pentagon tilings and even some 4-block pentagon tilings.

By now the original MG^2 circulation of information on the pentagon problem had grown and information was traveling to three continents. Another group of amateurs—11th year school children in New South Wales, Australia, had spent a week investigating the problem of discovering convex equilateral pentagons which tile and the ways in which they tile. Led by their teachers, George Szekeres and Michael Hirschhorn, they had made good progress on the problem. One particular equilateral pentagon was capable of a great variety of tilings (this was the special tile "case *6*" of Marjorie Rice's first correspondence with me). Hirschhorn had discovered many unusual tessellations using this pentagon, including two beautiful central tessellations with just six-fold rotational symmetry (Figure 12).

In the summer of 1977 I provided Marjorie with the article "The 81 Isohedral Tilings of the Plane" by Branko Grünbaum and G. C. Shephard. The paper contained careful sketches illustrating the 81 distinct types of tilings and I felt that Marjorie might find among these some whose tiles could be dissected into congruent pentagons, thus creating new pentagonal tilings. Throughout the fall, she continued her previous work and also utilized the Grünbaum-Shephard article in a far more sophisticated manner than I had anticipated. "The isohedral tilings by Grünbaum and Shephard were of much interest. I have copied them into a four page version I could more easily use." She had, in fact, represented each of the

FIGURE 12
Michael Hirschhorn's central tessellation by an equilateral pentagon, fashioned into an engraved silver pendant.

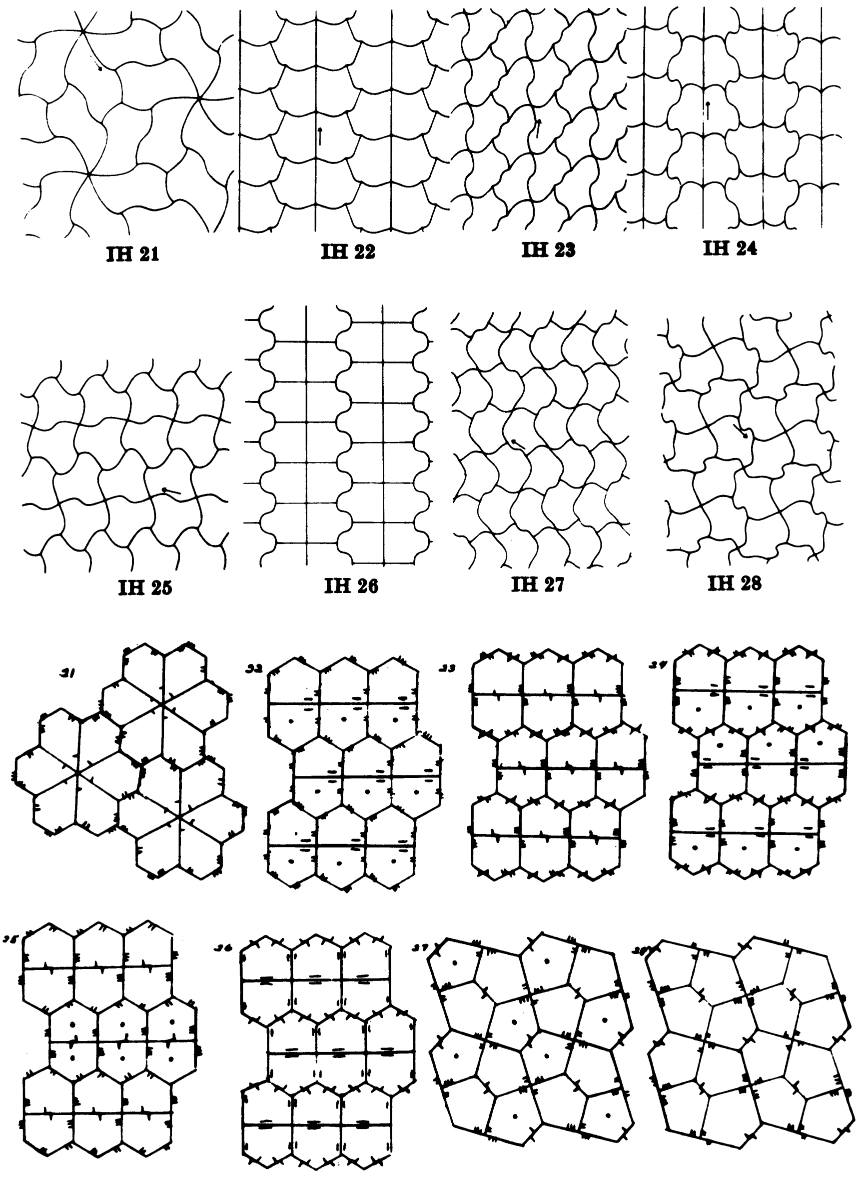


FIGURE 13

(Upper half) Several tilings by shaped tiles as they appear in "The 81 Types of Isohedral Tilings in the Plane", by Branko Grünbaum and G. C. Shephard (Mathematical Proceedings of the Cambridge Philosophical Society. September, 1977. pp. 190–191). (Lower half) The same tilings as redrawn in symbolic form by Marjorie Rice.

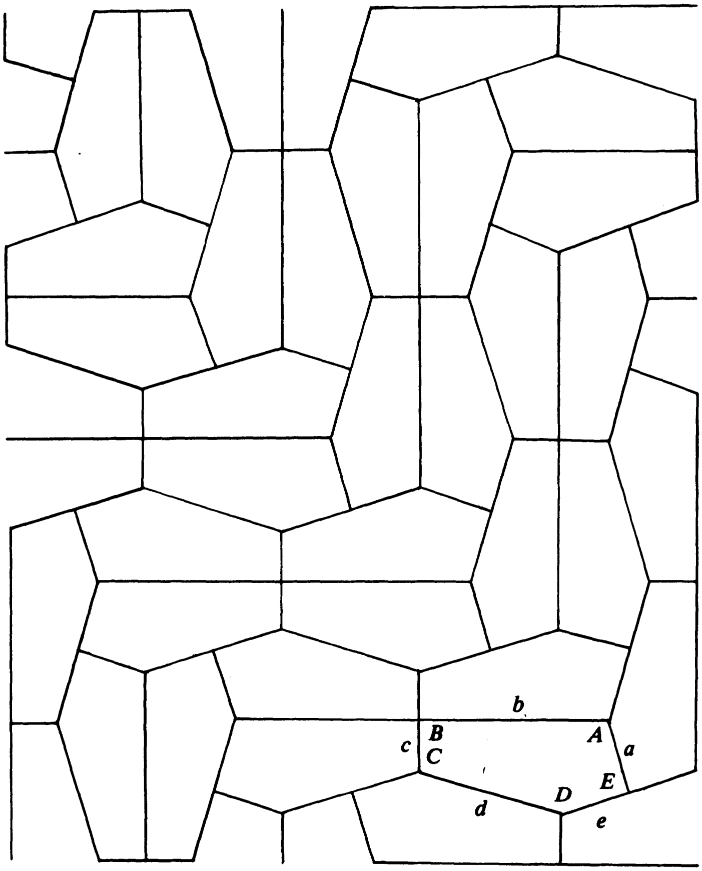
Grünbaum-Shephard tilings of shaped tiles by a symbolic marked tiling showing only the topological network of the tiling and the action of the symmetry group on the tiles (Figure 13). Using these, she had reanalyzed all of her previous tilings and discovered several new ones.

It was Christmas season, 1977, when she sent me a fat envelope containing her voluminous work. Again, there was a Christmas surprise. "Just a couple of weeks ago, this new type of pentagon turned up (and I thought there would be no more). This one (Figure 14) like type 4 has two opposite 90° angles but the requirements for the lengths of the sides are different." Her illustrative tiling by this new type 13 showed an interesting pattern of interlocked bow ties, made up of 4 of the pentagons. I had just received the galley proofs of my article on the pentagon problem and so was able to insert this latest discovery before publication. (The article, in the January 1978 issue of *Mathematics Magazine*, contains a fuller account of the mathematical details of pentagons which tile.) Still Marjorie's work on the problem did not end—she was determined to try to see if she could prove that all pentagons which could tile had been found. Although her attempt at proof was not complete, her thorough combinatorial check of all 2-block and 3-block patterns reduces the remaining task considerably. Perhaps by the time this story is published the question as to whether there are still other pentagons which tile will be answered. Hirschhorn is confident that all equilateral convex pentagons which tile have been found (these are described in the *Mathematics Magazine* article); his argument utilizes a computer in a proof by elimination of possible angle relationships.

What makes a person pursue a problem so steadfastly as Marjorie? She was not trained to do this, nor paid to do it, but obviously gained personal satisfaction in her patient and persistent search. No doubt her personal history is like that of many amateurs who are inspired by Gardner's writing.

She was born in 1923 in St. Petersburg, Florida, a first child. At age 5, she began school in Garden Valley School, a one-room country school with grades 1 through 8, having a total of about two dozen pupils. "My mother wished me to have a good start and had taught me well at home so I was placed in the second grade." She was a shy child, loved to read and "could easily become absorbed in a book or in my daydreams and forget the world around me." She did well in school; "arithmetic was easy and I liked to discover the reasons behind the methods we used." "I was interested in the colors, patterns and design of nature and dreamed of becoming an artist. ..." Her later years at the school "were enriched by two very fine teachers, Miss Keasey and Miss Timmons ... [who] helped make up for the deficiencies of a small country school."

"When I was in the 6th or 7th grade our teacher pointed out to us one day the Golden Section in the proportions of a picture frame. This immediately caught my imagination and though it was just a passing incident, I never forgot it. I've continued reading on a wide variety of



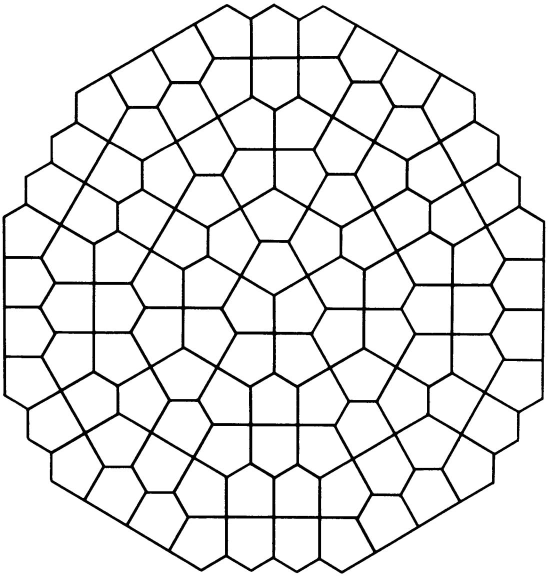
$B = E = 90°$
$2A + D = 360°$
$2C + D = 360°$
$a = e$
$a + e = d$

FIGURE 14

Tiling pentagon of type 13 discovered by Marjorie Rice in December, 1977.

subjects over the years and have been especially interested in architecture and the ideas of architects and planners such as Buckminster Fuller. I've come across the Golden Section again in my reading and considered its use in painting and design. A book that was especially helpful and inspiring to me in this regard was *The Geometry of Art and Life* by Matila Ghyka." Marjorie's interest in art continued—she became especially interested in textile design and the works of M. C. Escher. As she pursued the problem of pentagons and their tilings, she produced some beautiful geometric designs and imaginative Escher-like patterns (Figures 15, 16).

While in high school the family moved to Pine Castle, near Orlando,



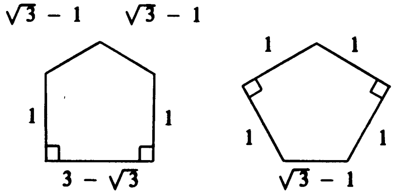


FIGURE 15

A beautifully symmetric tiling by Marjorie Rice which radiates outwards from the center and can be continued to fill the plane. Two pentagons, each having two 90° angles and three 120° angles, create the design.

Florida. At Orlando senior high Marjorie studied shorthand and typing to prepare for future employment and did poorly at both. She regretted that she could not take mathematics beyond the required general course. After graduating from high school at 16 she was employed first in the office of a laundry, then in a small printing firm until her marriage to Gilbert Rice in 1945. "During those years I frequented the public library and learned much about science, psychology and other subjects I had missed out on in school. I also started a correspondence course in commercial art. ..." After the Rices' first son was born they moved to San Diego, California. "These were very busy years for us both"—an understatement to be sure. The Rices raised five children and Gilbert started his own trade typesetting shop. Marjorie was drawn back into mathematics by her children. "When my oldest son, David, was in junior high, ... the 'new math' was just beginning to be used. ... I wanted to study his lessons and keep up with him

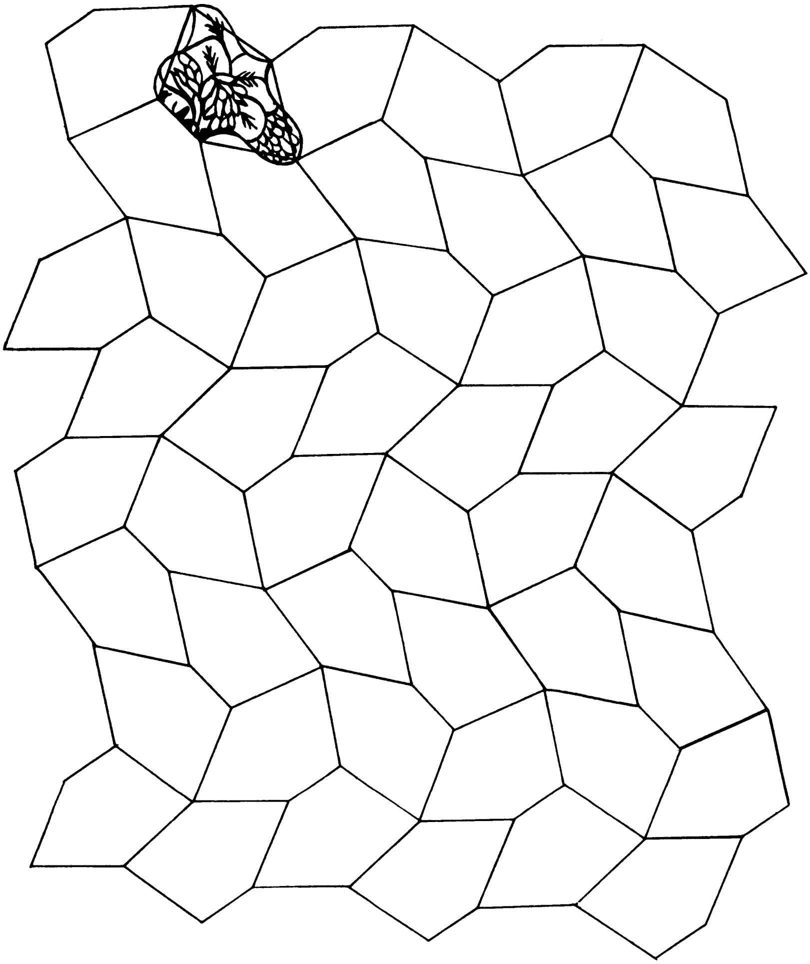


FIGURE 16A

Underlying grid for Bees in Clover (see Plate I.)

FIGURE 16

Three Escher-like plane-filling designs by Marjorie Rice are based on the geometric grids of some of her unusual tilings by pentagons. (M. C. Escher used a well-known tiling by pentagons as the underlying grid for some of his plane-filling designs.)

as he learned and he encouraged me to do this—but these were busy days. I soon fell behind and gave up the idea. My interest in his lessons continued however and I could often find solutions to his problems by unorthodox means, since I did not know the correct procedures. He shared with me the mathematical games he learned in class, such as Hex and three dimensional tic-tac-toe. ..."

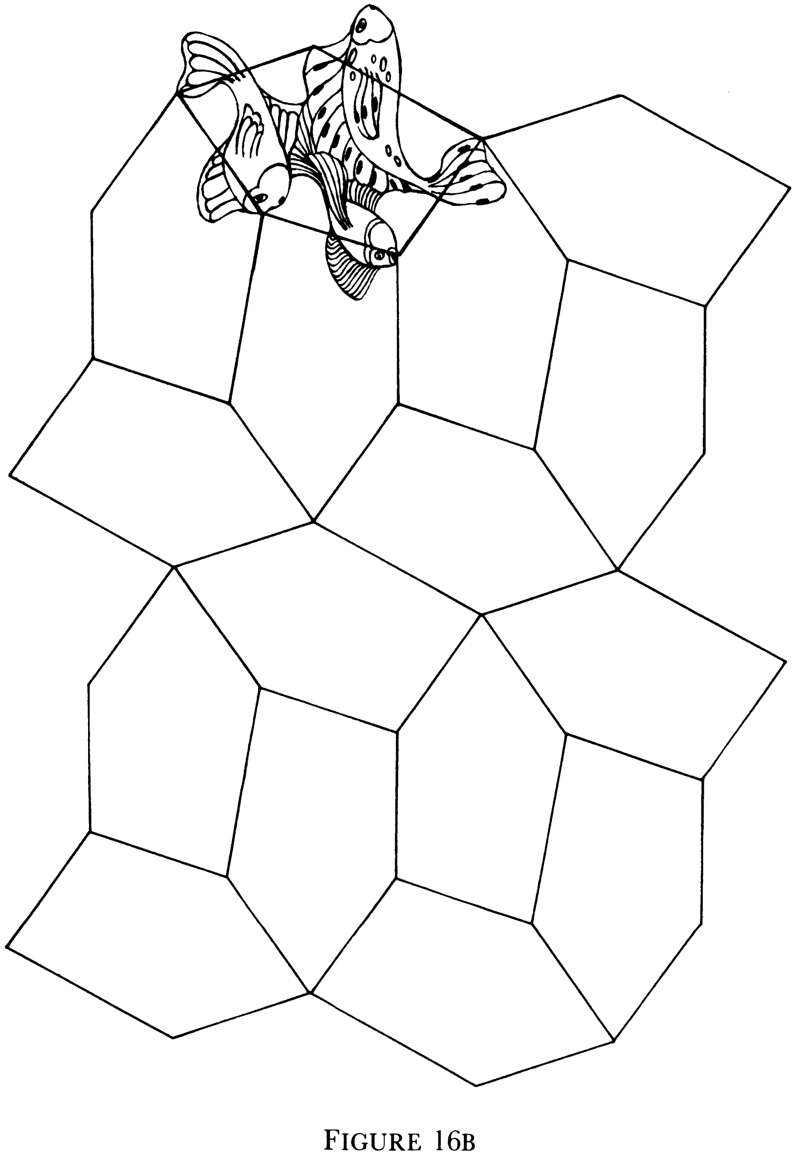


FIGURE 16B
Underlying grid for Fish (see Plate II.)

"I enjoy puzzles of all kinds, crosswords, jigsaw puzzles, mathematical puzzles and games, and have purchased books of mathematical puzzles over the years. Those of a geometric nature are a special delight. Thus when my son's *Scientific American* arrives I first turn to Martin Gardner's "Mathematical Games" section." Her absorbing fascination with such puzzles and keen perception of shapes, proportions and designs is tellingly revealed in her account of a recent trip. "In November, 1974, my husband and I started on a journey which would take us around the world. ... I had

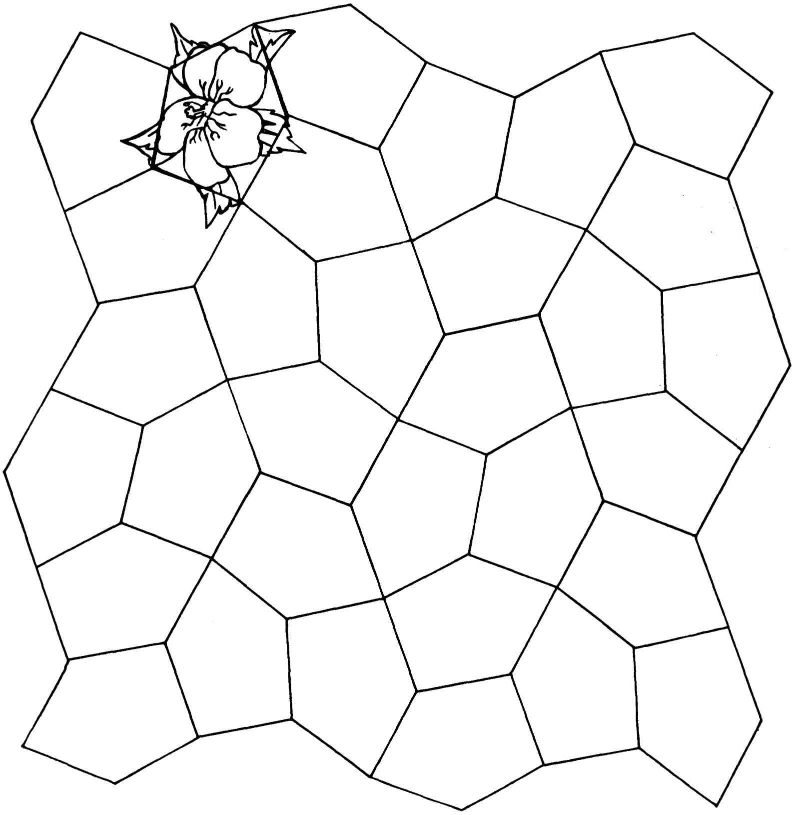


FIGURE 16C
Underlying grid for Hibiscus (see Plate III.)

much interest and curiosity concerning designs and proportions that were different and unfamiliar. I looked for such things wherever we traveled, taking notes of proportions of houses, doors, windows, division of space, the designs of grilles over windows. ... I especially enjoyed the delightful and colorful textile designs often in big bold patterns that were often worn in Ghana and Nigeria." "We kept our luggage to a minimum on this trip ... but I did slip in a small paperback book, *Work This One Out,* 105 puzzling brain-teasers by L. H. Longley-Cook. Thinking on these problems helped pass the time quickly when we had long periods of waiting. ..."

The mind and spirit are the forte of all such amateurs—the intense spirit of inquiry and the keen perception of all they encounter. No formal education provides these gifts. Mere lack of a mathematical degree separates these "amateurs" from the "professionals". Yet their dauntless curiosity and ingenious methods make them true mathematicians. Martin Gardner has awakened many such mathematicians.